UNIT 8 : RELATIONS

8.0 : Unit Objectives:

By the end of this Unit, learners should be able to:

- Explain Cartesian product of two sets.
- Characterised relations in terms of set theory .
- Understand matrix representation of relations.
- Describe and discuss different types of relations
- Describe reflexive, symmetric and transitive relations
- Describe equivalence relations
- Understand reflexive, symmetric and transitive closures

8.1 Unit Introduction:

Once the set of objects is introduced it is required to understand interrelationship among its elements. Associated with the relation is the act of comparing elements which are related to each other. In this unit we will first formalize the concept of a relation and then discuss types of relations. The fundamental requisite for the concept of relation is 'the Cartesian product of two sets'. Hence in this lesson we start with the revision of the concept of product of sets. We will also characterise relation in terms of set theory and represent a relation using matrix. Then we will discuss important class of relations which is equivalence relations. All these have useful applications in the design of digital computers and other sequential machines.

The notion of a relation between two sets of objects is quite common and basic concept. We use various relations in every day life as well as in mathematics. In every day life we always deal with human relationships. The word "relation" suggests some familiar examples of human relations such as "relation between a father and a daughter", "relation between brother and sister", "etc.

When we study relations, generally two sets are used and relation between their elements is considered.

Let M be the set of all men and W be the set of all women on the earth. Let "Ram" and "Rani" be the members of these two sets respectively. One can assume different relations between these two members as:

"Ram" *is father of* "Rani"

"Ram" *is a brother of* "Rani"

"Ram" *is husband of* "Rani"

"Ram" is *a son of* "Rani" etc.

In fact we can define relation between any two sets which are not necessarily of persons. Let A = {eggs, milk, rice, bread } and B = {cows, goats, hens} be two sets. Assume that if an element x of set A is produced by an element y of set B, then "x is related to y"; otherwise "x is not related to y".Then we observe that eggs and hens are related and milk and cows are related. But rice and hens are not related also bread and goats are not related.

We can define relation between elements of any one set also. In arithmetic there are many commonly used relations between any two numbers, few of such are “greater than”, “less than” or “equality” etc. In geometry we study the relation between the area of a circle and its radius, the relation between the volume of a cube and the length of its sides etc. We consider appropriate sets of objects on which these relations can be defined.

8.2 :Cartesian product of sets :

As mentioned above, every relation is essentially equal to a list of ordered pairs of some elements, and each ordered pair specifies that its first element is related to its second element. So while listing elements of two sets which are related we use pairs containing the first element of one set and the second element of the other set.

If three students Rani, Ram and Ganesh can offer either Mathematics or Biology for the examination, then we can list all possible choices of courses offered, by listing all possible pairs of the form (student, course). Obviously these pairs are ‘ordered pairs’ that means we consider the fixed sequence. We first consider the name of student and then a name of course he or she is offering. Then the following set of ordered pairs shows all possibilities of these two courses taken by these three students.

Choices = {(Rani, Mathematics), (Ram, Mathematics), (Ganesh, Mathematics),
(Rani, Biology), (Ram, Biology), (Ganesh, Biology)}.

This set named as “Choices” is a **product set** (or Cartesian product) of two sets which are

1. the set of students = {Rani, Ram, Ganesh } and
2. the set of courses = { Mathematics, Biology}

Definition 8.2.1: <u>Product set (or the Cartesian product)</u> : For any two nonempty sets A and B, the set of all ordered pairs (*a, b)*, where *a* is an element of *A* and *b* is an element of *B*, is called the Product set (or the Cartesian product) of A and B and is denoted by $A \times B$.

Thus, using the set builder form we can write, $A \times B = \{(a, b) / a \in A \text{ and } b \in B\}$.

The number of elements in $A \times B$ = the number of elements in set A $\times$ the number of elements in set B.

Examples:

- If A denotes the set of programmers and B denotes the set of computer languages, then $A \times B$ is a set of all possible ordered pairs (a, b) where $a \in A$ and $b \in B$. We can interpret this product set $A \times B$ as a set of all possible pairs of programmers and computer languages.
- Let L = {C, Pascal, COBOL} is a set of computer languages;

and S = {Windows, UNIX, dos} is a set of operating systems.

Then their product is

$L \times S$ = {(C, Windows), (Pascal, Windows), (COBOL, Windows), (C, UNIX), (Pascal, UNIX), (COBOL, UNIX), (C, dos), (Pascal, dos), (COBOL dos)}

- If A = {a, b} and B = {a, c, d} are two sets then the product sets are

  $A \times B$ = {(a, a), (a, c), (a, d), (b, a), (b, c), (b, d)}

  $B \times A$ = {(a, a), (a, b), (c, a), (c, b), (d, a), (d, b)}

Note that, product of sets is not a commutative operation, hence $A \times B \neq B \times A$ always.

8.3 Relations

A relation is a rule which assigns elements of one set to the elements of another set. Hence any set of ordered pairs defines a relation or a binary relation from one set to another set.

Definition 8.3.1: Relation : A relation R (or also called a binary relation R) from set A to set B is a subset R of the product set $A \times B$.

Examples:

- Consider the set, namely P = {(milk, cows), (eggs, hens)}, this set P is a relation as it is a set of ordered pairs. We can say that "a is related to b" if and only if "a is produced by b". In this case "a is produced by b" is a rule which assigns elements of set A = {eggs, milk, rice} to the elements of set B = {cows, hens}.
- Consider a set R = {(C, UNIX), (C, dos), (Pascal, dos), (COBOL, dos)}.

This set R is of ordered pairs and it is a relation from a set L = {C, Pascal, COBOL} to a set S = {Windows, UNIX, dos}. Where L is a set of computer languages and S is a set of operating systems. In this case an element 'x' from set L is related to an element 'y' from set S, if the computer language 'x' is used with the help of the operating system 'y'.

If $(a, b) \in R$ then it means that an element 'a' from set A is related to an element 'b' from set B by the relation 'R' and it is also denoted by 'aRb'. Also if $(x, y) \notin R$ then it means that an element 'x' from set A is not related to an element 'y' from set B by the relation 'R' and it is denoted by '$a \not{R} b$'.

The relation R can be described by any or all of the following three ways:

(i) listing all pairs belonging to set R or

(ii) by describing a membership rule for R or

(iii) by writing matrix of relation R or

If R is any relation from set A to set B, then the set

$\{a \in A \mid (a, b) \in R$ for some $b \in B\}$ is called as the **domain** of relation R.

And the set $\{b \in B \mid (a, b) \in R$ for some $a \in A\}$ is called as the **range** of relation R.

If A = B then R is called a relation **(**or a binary relation**) on** A, and in this case R is a relation such that $R \subset A \times A$. For a relation defined on set A the domain and range both are subsets of set A.

Examples:

- Let S be a set of students studying Computer science and N is a set of natural numbers between 101 to 120 representing Computers available for use. Let R = {(Rani, 101), (Ram, 105), (Ganesh, 103), (Geeta, 101)}. R is a set of ordered pairs in which the first elements are the names of students and the second elements are the numbers showing which computer is used by the student. Hence R is a relation from set S to the set N.

This relation R can be described in words by the rule:

'For $x \in S$ and $y \in N$, $(x, y) \in R$ if the student x is using the computer numbered y'.
For this relation S, the domain = {Rani, Ram, Ganesh, Geeta} and
the range = {101, 105, 103}.

- If A = {3, 5, 8, 15} and B= {5, 10, 15}, then

$$A \times B = \{(3,5), (3,10), (3,15), (5,5), (5,10), (5,15), (8,5), (8,10), (8,15), (15,5), (15,10), (15,15)\}.$$

(i) If S = { (5,5), (15 ,15)}, then S is a relation from set A to set B as $S \subset A \times B$.
This relation S can be described in words by the rule:

For $a \in A$ and $b \in B$, $(a, b) \in S$ if $a = b$.

For this relation S, the domain = {5, 15} and the range = {5, 15}.
(ii) If R = {(3 ,5), (3,10), (3,15), (5,10), (5,15), (8,15) }, then R is a relation from set A to set B as $R \subset A \times B$.

This relation R can be described in words by the rule:

For $a \in A$ and $b \in B$, $(a, b) \in R$ if $a < b$.

We observe that 3 is related to 5 by this relation but 8 is not related to 5 by this relation. For this relation R , the domain = {3, 5, 8} and the range = {5, 10, 15}

- If A = {a, b, c, d} and R is a relation on set A, where
R = {(a, a), (b, b), (b, c), (c, c), (c, b), (d, d)}.
Then R is a relation on set A as $R \subset A \times A$.

This R is a relation which can not be described by some obvious membership rule as in previous example.
For this relation R, the domain = {a, b, c, d } and the range = {a, b, c, d}.

***<u>Self Test I</u>*:**

In the Exercises 1 to 10, find the domain and range of the relation R. Also list all ordered pairs specifying the relation R, if it is not given as a set of ordered pairs.

1. A = {1, 2, 5 } and B = {1, 5, 9, 16} and relation R from A to B is, R = {(1,5), (1,9), (1,16), (2,5), (2,9), (2,16), (5, 9), (5,6) }.

2. A = B = {1, 2, 3, 4} and relation R from A to B is,

R = { (1, 1), (2, 2), (3, 3),(4, 4)}.

3. A = {1, 2, 3, 4, 5} and B = {1, 4, 9, 10, 16}. R is a relation from set A to set B defined as (a, b) $\in$ R if and only if b = a^2.

4. A = {1, 2, 3, 4} and B = {1, 4, 9, 16}. R is a relation from set A to set B defined as (a, b) $\in$ R if and only if b is a multiple of a.

5. A = {2, 4, 6, 8, 10} and B = {2, 3, 4, 5}. R is a relation from set A to set B defined as (a, b) $\in$ R if and only if a $>$ b.

6. A = {1, 2, 3, 4, 5} and R is a relation defined on set A as (a, b) $\in$ R if and only if b – a = 1.

7. A = {1, 2, 3, 4, 5, 6} and R is a relation defined on set A as (a, b) $\in$ R if and only if a + b = 9.

8. A = {3, 6, 9, 12, 15} and R is a relation defined on set A as (a, b) $\in$ R if and only if a + 3 = b.

9. A = { –2, –1, 0, 1, 3} and R is a relation defined on set A as (a, b) $\in$ R if and only if | a | $<$ | b |.

10. A = {1, 2, 3, 4} and R = {(1, 1), (1, 3), (3, 1), (2, 2), (3, 3), (4, 1), (4,4)}.

8. 4 Types of Relations:

In general when we define a relation R from a set A to a set B then every element of A need not be related to an element of set B. But if every element of A is related to some element of set B then there are different types of such relations. Sometimes such relations are also referred to as “correspondence”.

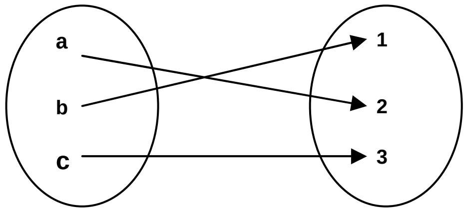
Definition 8.4.1: <u>One to one relation</u> : A relation R from set A to set B is a one to one relation if every element of A is related to a unique element of set B.

Examples:

- Let A = {1, 3, 7, 9} and B = {1, 9, 25, 49, 81, 100}. Define a relation R from set A to set B as (a, b) $\in$ R if and only if b = a^2.
  Then R as a set of ordered pairs can be written as
  R = { (1, 1), (3, 9), (7, 49), (9, 81)}
  We observe that every element of A is related to a unique element of set B, so this relation is one to one relation from set A to set B.
- Let A = {1, 3, 5, 7} and B = {1, 2, 3, …, 10 }. Define a relation R from set A to set B as (a, b) $\in$ R if and only if a = b.
  Then R as a set of ordered pairs can be written as
  R = { (1, 1), (3, 3), (5, 5), (7, 7) }

We observe that every element of A is related to a unique element of set B, so this relation is also one to one relation from set A to set B.

- Let A = {a, b, c} and B = {1, 2, 3 }. Define a relation R from set A to set B as R = { (a, 2), (b,1), (c,3)} this relation is by definition one to one relation. This can also be understood by the following Venn diagram representation.



One to one relation

Definition 8.4.2 : <u>One to many relation</u> : A relation R from set A to set B is a one to many relation if every element of A is related to some element of set B and atleast one element of A is related to two or more elements of set B.

Examples:

- Let A = {1, 3, 7, 9} and B = {1, 5, 15, 20 }. Define a relation R from set A to set B as (a, b) $\in$R if and only if a < b.

  Then R as a set of ordered pairs can be written as

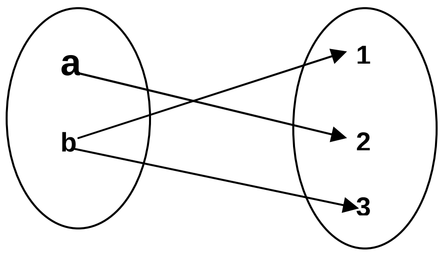
R = { (1, 5), (1,15), (1,20), (3, 5), (3, 15), (3, 20), (7, 15), (7, 20), (9, 15), (9, 20)}

  We observe that every element of A is related to two or more elements of set B, so this relation is one to many relation from set A to set B.

- Let A = {2, 4, 6, 8, 10} and B = {1, 2, 3, **…**, 9 }. Define a relation R from set A to set B as (a, b) $\in$R if and only if a + b > 10.

  By this relation every element of A is related to two or more elements of set B, so this relation is one to many relation from set A to set B.

- Let A = { a, b } and B = {1, 2, 3 }. Define a relation R from set A to set B as R = { (a, 2), (b,1), (b,3) } this relation is by definition one to many relation. This can be understood by the following Venn diagram representation.



One to many relation

Definition 8.4.3 : <u>Many to one relation</u> : A relation R from set A to set B is a many to one relation if every element of A is related to some element of set B and two or more elements of A are related to unique element of set B.

Example:

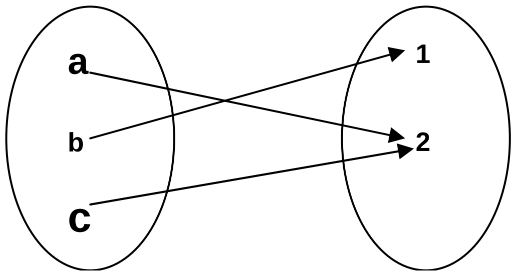
- Let A = { –2, –1, 1, 3 } and B = { 1, 4, 9, 25 }. Define a relation R from set A to set B as (a, b) $\in$R if and only if $a^2 = b$.

  Then R as a set of ordered pairs can be written as

R = { (–2, 4), (–1,1), (1,1), (3, 9) }

  We observe that two elements –1 and 1 of A are related to unique element 1 of set B, so this relation is many to one relation from set A to set B.

- Let A = { a, b, c } and B = { 1, 2 }. Define a relation R from set A to set B as R = { (a, 2), (b,1), (c,2) } this relation is by definition many to one relation. This can be understood by the following Venn diagram representation.



Many to one relation

Definition 8.4.4 : Many to many relation : A relation R from set A to set B is a many to many relation if every element of A is related to some element of set B and two or more elements of A are related to two or more elements of set B.

Examples:

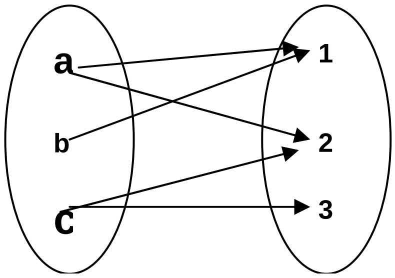
- Let A = {1, 3, 5, 7} and B = {1, 2, 3, **…**, 10 }. Define a relation R from set A to set B as (a, b) $\in$R if and only if a – b is even.

By this relation more than one elements of A are related to two or more elements of set B, so this relation is a many to many relation from set A to set B.

- Let A = {2, 4, 6, 8, 10} and B = {1, 2, 3, **…**, 9 }. Define a relation R from set A to set B as (a, b) $\in$R if and only if a + b is a multiple of 5.

By this relation more than one elements of A are related to two or more elements of set B, so this relation is also a many to many relation from set A to set B.

- Let A = { a, b, c } and B = {1, 2, 3 }. Define a relation R from set A to set B as R = { (a, 1), (a, 2), (b,1), (c,2), (c,3) } this relation is by definition many to many relation. This can be understood by the following Venn diagram representation.



Many to many

8. 5: Equivalence Relations and Equivalence Classes

When we define a relation on any set, then it satisfies different properties. According to some properties satisfied by relations we define different types of relations.

Definition 8.5.1: Reflexive relation: A relation R on a set A is a reflexive relation if every element in A is related to itself.

or R is a **reflexive** relation on A, if $(a, a) \in R$ for every $a \in A$.

Examples:

- Let A be the set of all male Americans and let R be a relation defined on A as $(a, b) \in R$ if 'a is a brother of b'. Then R is not a reflexive relation on A because any man can not be a brother of himself.
- If A = {1, 2, 3, 4} and
  R = {(1,1), (2,1), (3,1), (4,1), (4, 2), (3,3),(4,4)} then R is not a **reflexive** relation on set A because every element of A is not related to itself. Here we observe that, $(1, 1) \in R$, $(3, 3) \in R$, $(4, 4) \in R$ but $(2, 2) \notin R$.
- If A = {a, b, c} and R is a relation on set A, where
  R = {(a, a), (b , b), (b, c), (c, c), (c, b)}, then R is a **reflexive** relation on A because $(a, a) \in R$, $(b, b) \in R$, $(c, c) \in R$.
- Let L be the set of straight lines in a plane. The relation R is defined as line l_1 is related to line l_2 if line l_1 is parallel to line l_2. Then 'to be parallel' is a reflexive relation on L because every line in plane is always parallel to itself. So every element in set L is related to itself.
- The relation R on A = {1, 2, 3, 4} is defined by $(x, y) \in R$ if $x^2 \geq y$, then R is a reflexive relation as $x^2 \geq x$ for all $x \in A$, so $(x, x) \in R$ $\forall x \in A$.
  If we write the relation R as a set of ordered pairs, then we observe that, $(1, 1) \in R$, $(2, 2) \in R$, $(3, 3) \in R$, $(4, 4) \in R$.

Definition 8.5.2 : Symmetric relation : A relation R on a set A is a symmetric relation if, for all $a, b \in A$ if $(a, b) \in R$ then $(b, a) \in R$

or R is a symmetric relation on A if we have (a, b) ∈ R as well as (b, a) ∈ R .

Examples:

- Let A be the set of all male Americans and let R be a relation defined on A as (a, b) ∈R if 'a is a brother of b'. Then R is a symmetric relation on A because if x is a brother of y then y is a brother of x.
- If A = {a, b, c} and R is a relation on set A, where R = {(a, a), (b , b), (b, c), (c, c), (c, b)}, then R is a **symmetric** relation on A because (a, a) ∈R, (b, b) ∈ R , (c, c) ∈ R , (b, c) ∈R and (c, b) ∈ R.. So, whenever (x, y) ∈ R we observe that (y, x) ∈ R, for x, y ∈A.
- Let L be the set of straight lines in a plane. The relation R is defined as line l_1 is related to line l_2 if line l_1 is parallel to line l_2 If line l_1 is parallel to line l_2 then line l_2 is parallel to line l_1.Therefore if l_1 R l_2 then l_2 R l_1 for all such lines l_1 and l_2 in L. Hence R i.e. to be parallel is a symmetric relation on set L.
- If A = {1, 2, 3, 4} and R = {(1,1), (2,1), (3,1), (4,1),(2, 2),(4, 2), (3,3),(4,4)} then R is not a symmetric relation on A because here (2,1)∈R but (1, 2)∉ R.
- If A = {1, 2, 3, 4} and R = {(1,1), (2,1), (3,1), (4,1),(2, 2),(4, 2), (1, 2), (1, 3), (1, 4), (2, 4)}. This relation R on A is a **symmetric** relation because we observe that (y, x)∈R , for all x, y ∈A such that (x, y) ∈ R .

Definition 8.5.3: Transitive relation: A relation R on a set A is a transitive relation if for all a, b, c∈A, whenever (a, b) ∈ R and (b, c) ∈ R then (a, c)∈ R.

or

R is a transitive relation on A if we have (a, b)∈R, (b, c)∈R as well as (a, c)∈R.

Examples:

- Let A be the set of all male Americans and let R be a relation defined on A as (a, b) ∈R if 'a is a brother of b'. Then R is not a transitive relation on A because, x is a brother of y and y is a brother of x but x ix not a brother of x.
i.e. If (x, y) ∈R and (y, x) ∈R then it is not true that (x, x) ∈R .
- let A = {a, b, c} and R is a relation on set A, where R = {(a, a), (b, b), (b, c), (c, c), (c, b)}.

Here we observe that R satisfies the transitivity condition i.e. (x, z) ∈ R, for all x, y, z ∈A such that (x, y) ∈R and (y, z) ∈R

For (a, a) ∈R, consider x = a, y = a and z = a, these x, y, z satisfy the transitivity condition.

For (b, b) ∈R, consider x = b, y = b and z = b, these x, y, z satisfy the transitivity condition.

For (c, c) ∈R, consider x = c, y = c and z = c, these x, y, z satisfy the transitivity condition.

Also
(b, c) $\in$R, (c, b) $\in$R and (b, b)$\in$R.
(c, b) $\in$R, (b, c) $\in$R and (c, c) $\in$R.
Thus relation R on a set A is a **transitive relation.**

- Let L be the set of straight lines in a plane. The relation R is defined as line l_1 is related to line l_2 if line l_1 is parallel to line l_2 . If line l_1 is parallel to line l_2 and line l_2 is parallel to line l_3 then line l_1 is parallel to the line l_3. Therefore if l_1 R l_2 and l_2 R l_3 then l_1 R l_3 for the lines l_1, l_2 and l_3 in L. Therefore R i.e. 'to be parallel' is a transitive relation on set L.
- If A = {1, 2, 3, 4} and
  R = {(1,1), (2,1), (3,1), (4,1),(2, 2),(4, 2), (3,3),(4,4), (1, 3),(4, 3), (2,3)}, then R is a transitive relation on A, because here (x, z) $\in$ R, for all x, y, z $\in$A such that (x, y) $\in$R and (y, z) $\in$R .
- If A = {1, 2, 3, 4} and
  R = {(1,1), (2,1), (3,1), (4,1),(2, 2),(4, 2), (3,3),(4,4)} then R is a transitive relation on A because here (x, z) $\in$ R, for all x, y, z $\in$A such that (x, y) $\in$R and (y, z) $\in$R.

Definition 8.5.4 : <u>Equivalence relation</u>: A relation R on a set A is an equivalence relation if it is reflexive, symmetric and transitive relation.

or

A relation R on a set A is an **equivalence relation** if it satisfies the following three properties:
(i) (a, a) $\in$ R for all a $\in$A.
(ii) If (a, b) $\in$ R then (b, a) $\in$ R, for a, b $\in$A.
(iii) If (a, b) $\in$ R and (b, c) $\in$ R then (a, c) $\in$ R, for all such a, b, c $\in$A.
Examples:

- If A = {a, b, c} and R is a relation on set A, defined as a set of ordered pairs as R = {(a, a), (b , b), (b, c), (c, c), (c, b)}, then R is an **equivalence relation** on A because it is reflexive, symmetric and transitive relation on A.
- If L is the set of straight lines in a plane and the relation R is defined as o l_1 R l_2, if line l_1 is parallel to line l_2 , then R is an **equivalence relation** on L because it is reflexive, symmetric and transitive relation on L.
- Let A = {1, 2, 3, 4} and
  R = {(1,1), (2,1), (3,1), (4,1),(2, 2),(4, 2), (3,3),(4,4), (1, 3),(4, 3), (2,3)},
  R is a reflexive relation as, (1, 1)$\in$ R, (2, 2)$\in$R, (3, 3)$\in$R, (4, 4)$\in$R also
  R is a transitive relation on A, because here (x, z) $\in$ R, for all x, y, z $\in$A such that (x, y) $\in$R and (y, z) $\in$R.

But it is not an equivalence **relation** on A because it is not a symmetric relation.
R is not a symmetric relation as (4, 1) $\in$R but (1, 4)$\notin$ R.

Definition 8.5.5 : Equivalence class : If R is an equivalence relation defined on set A, and if $a \in A$ is any element of A, then equivalence class of a is defined as the set of all elements of A which are related to a. It is denoted as $\bar{a}$ or [a].

Using Set theory we can write, $\bar{a} = [\, a\,] = \{x \in A \;/\; (a, x) \in R\}$

Example:

- If A = {1, 2, 3, 4} and

R = {(1,1), (1, 3), (3,1), (3,3), (2, 2),(4, 4)} then R is an equivalence relation on A. Then by definition the equivalence class of 1 is defined as the set of all elements of A which are related to 1. Here 1 is related by R to 1 and 3.

$\therefore\; \bar{1} = [\,1\,] = \{1, 3\}$.

Similarly $\bar{2} = \{2\}$, $\bar{3} = \{1, 3\}$ and $\bar{4} = \{4\}$.

- If A = {a, b, c, d} and R is a relation on set A, defined as a set of ordered pairs as R = {(a, a) , (b , b), (b, c), (c, c), (c, b), (a, d),(d, a), (d, d) }, then R is an equivalence relation on A. The equivalence class of a is defined as the set of all elements of A which are related to a. Here a is related by R to a and d.

$\therefore\; \bar{a} = \{a, d\}$.

Similarly $\bar{b} = \{b, c\}$, $\bar{c} = \{b, c\}$, and $\bar{d} = \{a, d\}$.

❖ *Self Test II*: Select the correct alternative from the given alternatives.

1. If A = {1, 10, 100, 1000 } and B = {1, 4, 9, 16}, then How many elements are there in the product set $A \times B$?

(a) 11 **(b)** 12 **(c)** 9 **(d)** 16.

2. On set A = {1, 2, 3, 4} define relation R = {(1, 1), (1, 2), (3,1), (3, 2), (3, 3), (3, 4), (4,3), (4,4)}. What is the domain of relation R?

(a) {1, 4} **(b)** {1, 2, 3} **(c)** {1, 3, 4} **(d)** {1, 2, 3, 4}.

3. On set A = {1, 2, 3, 4} define relation R = { (1, 1), (1, 2), (1, 4), (2, 2), (2, 4), (3, 4), (4,4)}. What is the range of relation R?

(a) {1, 2, 4} **(b)** {1, 2, 3} **(c)** {1, 3, 4} **(d)** {1, 2, 3, 4}.

4. What is an equivalence relation?

(a) A relation which is reflexive and symmetric.

(b) A relation which is symmetric and transitive.

(c) A relation which is reflexive, symmetric and transitive.

(d) A relation which is reflexive and transitive but not symmetric.

5. Let A = {1, 2, 3, 4, 5} and R is a relation defined on set A as $(a, b) \in R$ if and only if a + b = 10. Then relation R is of what type?

(a) a reflexive relation **(b)** not a reflexive relation

(c) not a symmetric relation **(d)** an equivalence relation.

6. Let A = {1, 2, 3, 4, 5} and R is a relation defined on set A as $(a, b) \in R$ if and only if $a \geq b$. Then relation R is of what type?

(a) a symmetric relation **(b)** not a reflexive relation

(c) a reflexive and transitive relation **(d)** an equivalence relation.

7. Let A = {1, 2, 3, 4, 5} and R is a relation defined on set A as aRb if and only if a – b is even. Then relation R is of what type?

(a) a reflexive relation **(b)** not a reflexive relation
(c) not a symmetric relation **(d)** an equivalence relation.

8. On set A = {a, b, c} define relation R = {(a, a), (b, b), (c, c)}. Then relation R is of what type?

(a) only a reflexive relation **(b)** not a reflexive relation
(c) not a symmetric relation **(d)** an equivalence relation.

9. On set A = {a, b, c} define relation R = {(a, b), (b, a), (c, c)}. Then relation R is of what type?

(a) only a reflexive relation **(b)** not a reflexive relation
(c) not a symmetric relation **(d)** an equivalence relation.

10. On set of all triangles in a plane, define a relation S as two triangles ΔABC and ΔPQR are related by S if and only if they are congruent triangles. Then relation S is of what type?

(a) only a reflexive relation **(b)** not a reflexive relation
(c) not a symmetric relation **(d)** an equivalence relation.

8. 6 : Matrix of a Relation

When the number of elements in a set is more, the listing of relation on it in the form of ordered pairs is not easy. In such case it is more easy to use matrix representation of relation. It is useful for analysis of a relation by a computer.

If A and B are two finite sets containing m and n elements respectively and R is a relation from set A to set B , then the matrix of relation R is a matrix of order $m \times n$ containing entries as 0s and 1s only. It is the matrix in which rows correspond to the elements of set A and the columns correspond to the elements of set B. If $a \in A$ and $b \in B$ are two arbitrary elements and if $(a, b) \in R$, then the entry in the row corresponding to the element ‘a’ and column corresponding to the element ‘b’ is equal to 1 and it is equal to 0 otherwise.

Note that :

1. The matrix on relation on set a is a square matrix. If orders of elements in the set A and /or B are changed then the Matrix of relation R is different.

2. The relation R is a reflexive relation if and only if the matrix of R has 1’s on the main diagonal.

3. The relation R is a symmetric relation if and only if the matrix of R is a symmetric matrix.

4. It is not easy to decide whether R is a transitive relation by observing the matrix.

Examples:

- If A = { 3, 5, 8, 15 } and B= {5, 10, 15 }, and S is a relation from set A to set B defined as S = { (5,5), (15 ,15)} .

The matrix of relation S is a matrix of order 4 × 3. It is the matrix in which rows correspond to the elements 3, 5, 8 and 15of set A (in this ordering); and the columns correspond to the elements 5, 10 and 15of set B (in this ordering).

In the matrix of relation S the entry in the row corresponding to the element '5' and the column corresponding to the element '5' is equal to 1 because (5, 5) $\in$S. Also (15, 15) $\in$S, therefore the entry in the row corresponding to the element '15' and the column corresponding to the element '15' is equal to 1. All other entries in the matrix are equal to 0.

Then the matrix of relation S for the given ordering of A and B is :

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

<u>Note that</u>: If the ordering of elements in A and B is changed then the places of 0s and 1s in the corresponding matrix of relation are changed.

- If A = { 3, 5, 8, 15 } and B= {5, 10, 15 }, and R is a relation from set A to set B defined as R = {(3 ,5), (3,10), (3,15), (5,10), (5,15), (8,15) }

Then the matrix of relation R = $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

- If A = {1, 2, 3, 4} and R is a relation on set A listed as a set R = {(1,1), (2,1), (3,1), (4,1),(2, 2),(4, 2), (3,3),(4,4)}.

Then the matrix of relation R is :

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Here we observe that R is a reflexive relation and the matrix of R has 1's on the main diagonal.

- The relation R on A = {1, 2, 3, 4} is defined by:

  (x, y) $\in$ R if $x^2 \geq y$.

This relation R can be written as a set of ordered pairs as:

R = {(1, 1), (2, 1), (3, 1), (4, 1), (2, 2), (3, 2), (4, 2), (3, 3), (3, 4), (4, 4)}

Then the matrix of relation R is:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

Here also R is a reflexive relation and the matrix of R has 1's on the main diagonal.

- If A = {a, b, c} and R is a relation on set A, where

R = {(a, a), (b, b), (b, c), (c, c), (c, b)}. Then the matrix of relation R is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Note that:

(i) R is a reflexive relation and the matrix of R has 1's on the main diagonal

(ii) R is symmetric and the matrix of R is also symmetric about its main diagonal.

❖ *Self Test III:*

In the Exercises 1 to 10, find the the matrix of the given relation R.

1. Set A = {1, 2, 5} and B = {1, 5, 9, 16} and relation R from A to B is,
R = {(1,5), (1,9), (1,16), (2,5), (2,9), (2,16), (5, 9), (5,6) }.

2. Set A = B = {1, 2, 3, 4} and relation R from A to B is,
R = { (1, 1), (2, 2), (3, 3),(4, 4)}.

3. Set A = {1, 2, 3, 4, 5} and B = {1, 4, 9, 10, 16}. R is a relation from set A to set B defined as (a, b) $\in$ R if and only if $b = a^2$.

4. Set A = {1, 2, 3, 4} and B = {1, 4, 9, 16}. R is a relation from set A to set B defined as (a, b) $\in$ R if and only if b is a multiple of a.

5. Set A = {2, 4, 6, 8, 10} and B = {2, 3, 4, 5}. R is a relation from set A to set B defined as (a, b) $\in$ R if and only if $a > b$.

6. Set A = {1, 2, 3, 4, 5} and R is a relation defined on set A as
(a, b) $\in$ R if and only if $b - a = 1$.

7. Set A = {1, 2, 3, 4, 5, 6} and R is a relation defined on set A as
(a, b) $\in$ R if and only if $a + b = 9$.

8. Set A = {3, 6, 9, 12, 15} and R is a relation defined on set A as
(a, b) $\in$ R if and only if $a + 3 = b$.

9. Set A = { –2, –1, 0, 1, 3} and R is a relation defined on set A as
(a, b) $\in$ R if and only if $|a| < |b|$.

10. Set A = {1, 2, 3, 4}
and R = {(1, 1), (1, 3), (3, 1), (2, 2), (3, 3), (4, 1), (4,4)}.

8. 7 Closure of relation

Definition 8.5.1: Closure of relation : R Let R be a relation on set A**.** The smallest relation R^* on the set A that contains R as a subset and which possesses the desired property is called the **Closure** of relation R with respect to the property under consideration.

There are different closures of R as below:

(1) Reflexive Closure: Let R be a relation on set A**.** The reflexive Closure of R is the smallest reflexive relation on A that contains R as a subset.

Examples:

- Set A = {1, 2, 3, 4} and relation R defined on A is, R = { (1, 1), (2, 2), (3, 3) },
  Then its reflexive closure R* = { (1, 1), (2, 2), (3, 3), (4, 4) }.
- Set A = {1, 2, 3, 4} and R = {(1, 1), (3, 1), (4, 1), (4,4)}.

Then its reflexive closure R* {(1, 1), (1, 3), (3, 1), (2, 2), (3, 3), (4, 1), (4,4)}.

(2) Symmetric Closure: Let R be a relation on set A**.** The symmetric Closure of R is the smallest Symmetric relation on A that contains R as a subset.

Examples:

- Set A = {1, 2, 3, 4} and relation R defined on A is, R = { (1, 1), (2, 2), (3, 3) },
  Then its symmetric closure R* = R ={ (1, 1), (2, 2), (3, 3) }.
- Set A = {1, 2, 3, 4} and R = { (1, 1), (1, 3), (2, 2), (4, 1), (4,4)}

Then its reflexive closure R* { (1, 1), (1, 3), (3, 1), (2, 2), (4, 1), (4, 1), (4,4)}.

(3) Transitive Closure: Let R be a relation on set A**.** The transitive Closure of R is the smallest Transitive relation on A that contains R as a subset.

Examples:

- Set A = {1, 2, 3, 4} and relation R defined on A is, R = { (1, 1), (2, 2), (3, 3) },
  Then its transitive closure R* = R ={ (1, 1), (2, 2), (3, 3) }.
- Set A = {1, 2, 3, 4} and R = { (1, 1), (1, 3), (2, 4), (4, 2), (4,4) }

Then its reflexive closure R* = { (1, 1), (1, 3), (2, 2), (4, 2), (2, 4), (4,4) }.

❖ *Self Test IV:*

Exercise 1 : Select the correct alternatives from the given.

1. If R is a relation on set A**.** Then what is the smallest reflexive relation on A that contains R as a subset called?

a) Transitive closure of R **b)** Symmetric closure of R.
c) Reflexive closure of R. **d)** Partial closure of R

2. If R is a relation on set A. Then what is the smallest symmetric relation on A that contains R as a subset called?
   - **a)** Transitive closure of R
   - **b)** Symmetric closure of R.
   - **c)** Reflexive closure of R.
   - **d)** Partial closure of R
3. If R is a relation on set A. Then what is the smallest transitive relation on A that contains R as a subset called?
   - **a)** Transitive closure of R
   - **b)** Symmetric closure of R.
   - **c)** Reflexive closure of R.
   - **d)** Partial closure of R

Exercise 2: Find the reflexive, symmetric and transitive closures of each of the following by observation .

1. Let A = {1, 2, 5} and B = {1, 5, 9, 16} and relation R from A to B is,
   R = { (1,5), (1,9), (1, 16), (2,5), (2,9), (2,16), (5, 9), (5, 16) }.
2. Let A = {1, 2, 3, 4} and R is a relation on set A,
   R = {(1, 1), (2, 2), (3, 3),(4, 4)}.
3. Let A = {1, 2, 3, 4, 5} and B = {1, 4, 9, 10, 16}. R is a relation from set A to set B defined as (a, b) $\in$ R if and only if $b = a^2$.
4. Let A = {1, 2, 3, 4} and B = {1, 4, 9, 16}. R is a relation from set A to set B defined as (a, b) $\in$ R if and only if b is a multiple of a.
5. Let A = {2, 4, 6, 8, 10} and B = {2, 3, 4, 5}. R is a relation from set A to set B defined as (a, b) $\in$ R if and only if $a > b$.

8.8 Summary for Unit 8

In this unit learners studied the following topics in details:

1. Revised the concept of the Cartesian product of the two sets.
2. Definition and idea of a relation or a binary relation R from set A to set B
3. Different types of relations such as one to one, many to one and many to many relations.
4. Different types of relations such as reflexive, symmetric and transitive relations and equivalence relations.
5. How to represent a relation by a matrix.
6. The concept of the Closure of relation R and different types of closures.