UNIT 10 : Vectors

10.0 : Unit Objectives:

By the end of this Unit, learners should be able to:

- Understand concept of vectors.
- Describe and discuss different types of vectors.
- Perform different operations on vectors.
- Compute dot product, cross product, scalar triple product of vectors.
- Identify Collinear and coplanar vectors.

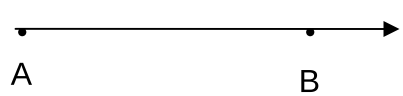
10.1 : Unit Introduction:

Vectors are fundamental objects in the study of physical sciences. They can be used to represent any quantity that has both a magnitude and direction, such as velocity, the magnitude of which is speed. Another quantity represented by a vector is force since it has a magnitude and direction. Vectors are also used to describe many other physical quantities, such as displacements, acceleration, momentum, angular momentum etc. Electric field and magnetic field are also represented as a system of vectors at each point of a physical space. In this unit we will study the definition, examples, different types and properties of vectors.

10.2 : Vectors:

The physical quantities which have only magnitude and are independent of direction are called scalar quantities. But there are some quantities which need direction along with the magnitude to describe them completely. The physical quantities which are described by, both magnitude and direction are called as vector quantities. Real numbers are scalars and displacements, velocity, acceleration etc. are vector quantities.

Definition 10.2.1: Vector: A vector is a geometric object that has both, a magnitude or length and a direction. A vector is frequently represented by a directed line segment i.e. by an arrow. A directed line segment AB with an initial point *A* and with a terminal point *B*, represents the vector $\overrightarrow{AB}$.

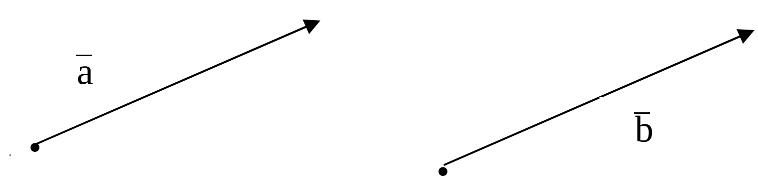


Definition 10.2.2: The magnitude (or modulus) of the Vector: The magnitude (or modulus) of the vector $\overrightarrow{AB}$ is the length of the segment AB and it is denoted as $|\overrightarrow{AB}|$.

Note that: Instead of denoting vectors by the initial and terminal vertices as $\overrightarrow{AB}$, $\overrightarrow{PQ}$ etc, vectors are also denoted as $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{p}$, $\bar{q}$ etc.

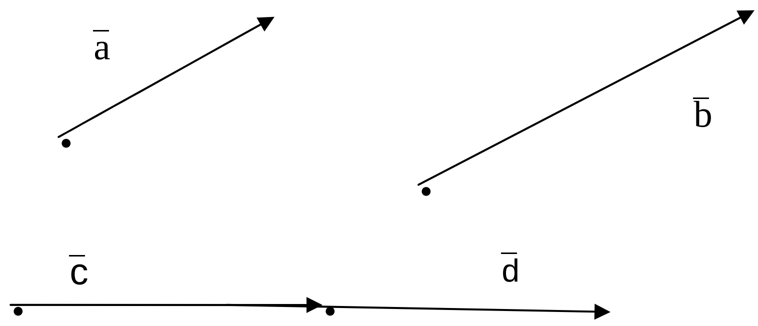
10. 3: Types of vectors:

1. Equal vectors: The vectors having same magnitude and same direction are said to be equal vectors.
   The two vectors $\bar{a}$ and $\bar{b}$ are said to be equal i.e. $\bar{a} = \bar{b}$ if $|\bar{a}| = |\bar{b}|$ and $\bar{a}$, $\bar{b}$ have same direction vectors.
   In the following diagrams $\bar{a}$ and $\bar{b}$ are equal vectors.



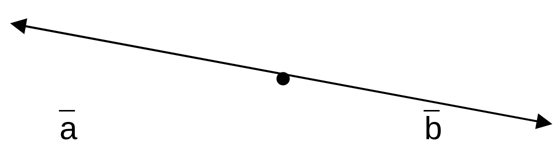
2. Unit Vector: A vector whose magnitude is one unit is said to be unit vector. A unit vector in the direction of the vector $\bar{a}$ is denoted by $\hat{a}$ (It is read as a cap). Obviously $\hat{a} = \dfrac{\bar{a}}{|\bar{a}|}$.
   In geometry the unit vectors along x, y and z axis are denoted by $\hat{i}, \hat{j}$ *and* $\hat{k}$ respectively.

3. Zero Vector or null vector: A vector whose magnitude is zero unit is said to be unit vector (or null vector). Obviously for a zero vector the initial point and the terminal point coincides.
4. Collinear vectors: Vectors which are parallel to the same line are called collinear vectors. In the following diagram vectors $\bar{a}$ and $\bar{b}$ are collinear vectors and also vectors $\bar{c}$ and $\bar{d}$ are collinear vectors.



5. <u>Coplanar vectors:</u> Vectors which are lying in the same plane or in the parallel planes are called coplanar vectors.

6. <u>Negative of a vector:</u> A vector having the same magnitude but opposite direction is the negative of the given vector. In the following diagram vectors $\bar{a}$ and $\bar{b}$ are negative vectors of each other. That is $\bar{b} = -\bar{a}$ and $\bar{a} = -\bar{b}$.



10.4 : Algebra of vectors:

As a vector is considered as a geometrical object, there are some algebraic operations defined on the set of vectors. These operations are addition, subtraction and scalar multiplication of vectors. After studying how to perform these operations, we will study the concept of position vector and how to represent a point in plane or in space in terms of vectors. This is needed because it is easy to deal with vector problems using position vectors.

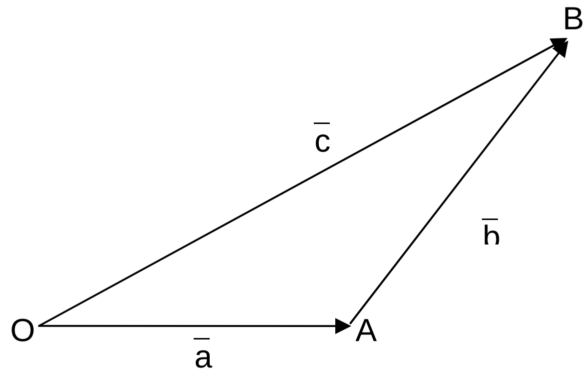
10.4.1 : Addition of vectors:

1. Triangle law of vector addition : If two vectors are such that initial point of the second vector is the terminal point of the first vector then triangle law of vector addition is used for their addition.

The law is stated as:

If O,A, B are three points such that $\overrightarrow{OA} = \bar{a}$ and $\overrightarrow{AB} = \bar{b}$ then the vector $\overrightarrow{OB} = \bar{c}$ is called the addition of the vectors $\overrightarrow{OA}$ and $\overrightarrow{AB}$

i.e. $\bar{c} = \bar{a} + \bar{b}$ or $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$



Note that:

(i) If A, B and C are any three non collinear points, then using triangle law of vector addition ,we have $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.

(ii) Using the above result for any two points A and B we can say that ,

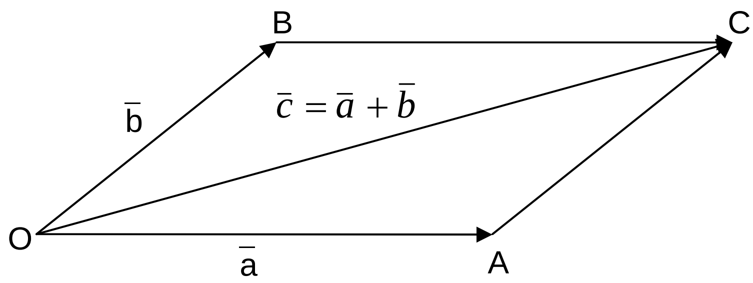
$\overrightarrow{AB} + \overrightarrow{BA} = \overrightarrow{AA} = \bar{0}$. $\therefore$ $\overrightarrow{AB} = -\overrightarrow{BA}$.

2. Parallelogram law of vector addition : If two vectors are such that initial point of the first vector is same as the initial point of the second vector then parallelogram law of vector addition is used for their addition.

The law is stated as:

If O,A, B, C are four points such that $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are adjacent sides of a parallelogram , then the diagonal of the parallelogram passing through common initial point represents the addition of theses two vectors i.e. $\overrightarrow{OA} + \overrightarrow{OB}$.

If $\overrightarrow{OA} = \bar{a}$ and $\overrightarrow{OB} = \bar{b}$ then the vector $\overrightarrow{OC} = \bar{a} + \bar{b}$ the addition of the vectors $\overrightarrow{OA}$ and $\overrightarrow{AB}$ as shown in the diagram below:



Properties of addition of vectors:

(i) Addition of vectors is commutative i.e. for any two vectors $\bar{a}$ and $\bar{b}$,
$\bar{a} + \bar{b} = \bar{b} + \bar{a}$.

(ii) Addition of vectors is associative i.e. for any three vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$,
$\bar{a} + (\bar{b} + \bar{c}) = (\bar{a} + \bar{b}) + \bar{c}$.

(iii) For any vector $\bar{a}$, $\bar{a} + \bar{0} = \bar{0} + \bar{a} = \bar{a}$.

(iv) For any vector $\bar{a}$, there exists the negative vector $-\bar{a}$ such that

$$\bar{a} + (-\bar{a}) = \bar{0}.$$

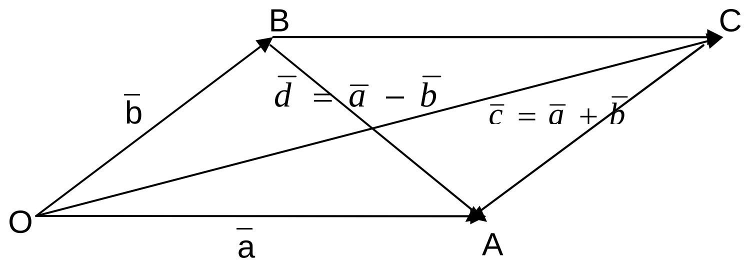
10.4.2 : Subtraction of vectors:

Definition 10.4.1: Subtraction of vectors: For any two vectors $\bar{a}$ and $\bar{b}$ their difference $\bar{a} - \bar{b}$ is defined as, $\bar{a} - \bar{b} = \bar{a} + (-\bar{b})$.

If O,A, B, C are four points such that $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are adjacent sides of a parallelogram , then by the parallelogram law of vector addition, the addition of these two vectors i.e. $\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC} = \bar{a} + \bar{b}$ where $\overrightarrow{OA} = \bar{a}$ and $\overrightarrow{OB} = \bar{b}$. Then we also observe in the parallelogram drawn below that,

$\overrightarrow{BA} = \overrightarrow{BC} + \overrightarrow{CA}$ …. by triangle law of vector addition.

Now $\overrightarrow{BC} = \overrightarrow{OA} = \bar{a}$, as these vectors have the same magnitude and the same direction; and $\overrightarrow{CA} = \overrightarrow{BO} = -\overrightarrow{OB} = -\bar{b}$. $\therefore$ $\overrightarrow{BA} = \bar{d} = \bar{a} - \bar{b}$.



***Self Test I*:**

10.4.3 : Scalar Multiplication of vectors:

Definition 10.4.2: Scalar Multiplication of vectors: For any vector $\bar{a}$ and any scalar (i.e a real number) k, scalar multiplication is defined as, the vector k $\bar{a}$.

Note that:

(i) The vectors $\bar{a}$ and k $\bar{a}$ are always collinear vectors.

(ii) Also two collinear vectors are scalar multiples of each other.

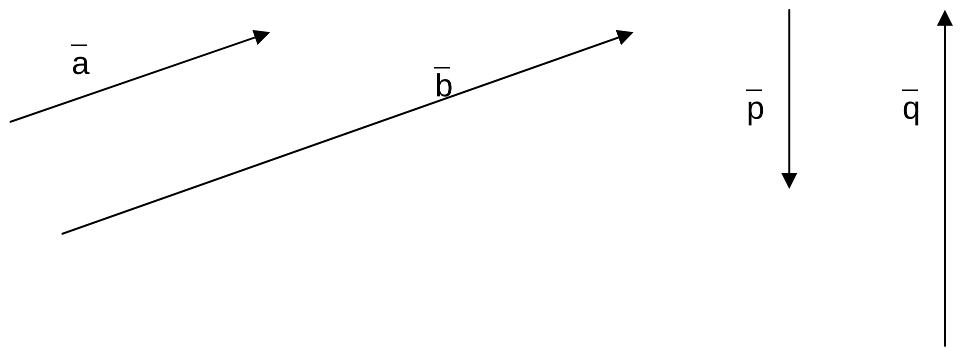
In the following diagram $\bar{a}$ and $\bar{b}$ are scalar multiples of each other and $\bar{p}$ and $\bar{q}$ are scalar multiples of each other.

Clearly | $\bar{b}$ |= 2| $\bar{a}$ | and $\bar{a}$ and $\bar{b}$ have the same direction.

$\therefore$ $\bar{b} = 2\,\bar{a}$ or equivalently $\bar{b} = \frac{1}{2}\,\bar{a}$.

And | $\bar{q}$ |= 2 | $\bar{p}$ | and $\bar{p}$ and $\bar{q}$ have the opposite directions.

$\therefore$ $\bar{q} = -2\,\bar{p}$ or equivalently $\bar{q} = -\frac{1}{2}\,\bar{p}$.



<u>Properties of scalar multiplication</u> <u>:</u>

If $\bar{a}$ and $\bar{b}$ are any two vectors and m,n are any real numbers then

(i) Scalar multiplication of a vector is associative i.e. $m(n\,\bar{a}) = m\,n(\bar{a})$ and also $m(n\,\bar{a}) = mn(\bar{a}) = n(m\,\bar{a})$.

(ii) Scalar multiplication is distributive over vector addition i.e.

$$m(\bar{a} + \bar{b}) = m\,\bar{a} + m\,\bar{b}.$$

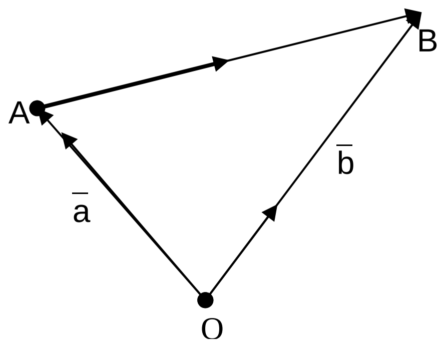
(iii) Addition is distributive over scalar multiplication of a vector i.e.

$$(m + n)\,\bar{a} = m\,\bar{a} + n\,\bar{a}\ .$$

10.4.4 : Representation of a point in plane using vectors :

1. <u>Position vectors of a point:</u> If P is any point and O is some fixed point, then the vector $\overrightarrow{OP}$ is called position vector of point P with respect to O. In particular if P is a point in plane then the Cartesian coordinates (x, y) of point P determine the position of point P uniquely. Hence the directed line segment $\overrightarrow{OP}$ is called position vector of point P ,with reference to origin O. If $\overrightarrow{OP}$ is denoted by $\bar{p}$ then the notation P($\bar{p}$)is used to denote that the point P is with position vector $\bar{p}$.

2. Representation of any vector in terms of position vectors:



Let A and B be two points with position vectors $\bar{a}$ and $\bar{b}$ respectively that means for some fixed point O, $\overrightarrow{OA} = \bar{a}$ and $\overrightarrow{OB} = \bar{b}$ as shown in above diagram. The by triangle law of vector addition, $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$.

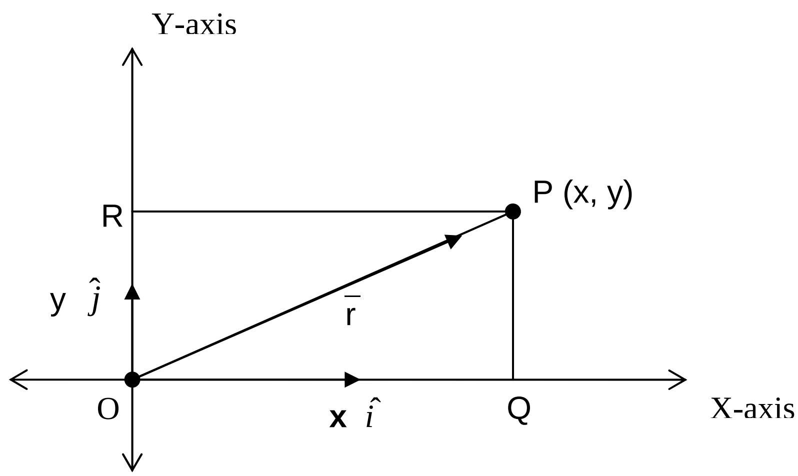
$$\therefore \quad \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} .$$

$$\therefore \quad \overrightarrow{AB} = \bar{b} - \bar{a} .$$

Thus any vector $\overrightarrow{AB}$ can be written as the difference of position vector of point B and position vector of point A.

Representation of a vector in plane: Let P be a point in plane with the Cartesian coordinates (x, y). Let O denotes the origin in the Cartesian plane and $\hat{i}, \hat{j}$ be unit vectors along X -axis and Y- axis respectively. Then as per the diagram given below, $\overrightarrow{OQ} = \mathbf{x}\,\hat{i}$ and $\overrightarrow{OR} = y\,\hat{j}$ as these two vectors collinear with the unit vectors $\hat{i}$ *and* $\hat{j}$ respectively. Now by parallelogram law of vector addition $\overrightarrow{OP} = \overrightarrow{OQ} + \overrightarrow{OR} = \mathbf{x}\,\hat{i} + y\,\hat{j}$

$$\therefore \bar{r} = \mathbf{x}\,\hat{i} + y\,\hat{j} .$$



Thus for every point P(x, y) in the Cartesian plane , there exists a unique vector representing it, which is $\overrightarrow{OP} = \bar{r} = x\hat{i} + y\hat{j}$.

The magnitude of this vector is, $|\overrightarrow{OP}| = |\bar{r}| = \sqrt{x^2 + y^2}$, by distance formula.

Similarly it can be explained that, for every point P(x, y, z) in the 3-D space , there exists a unique vector representing it, which is
$\overrightarrow{OP} = \bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

The magnitude of this vector is, $|\overrightarrow{OP}| = |\bar{r}| = \sqrt{x^2 + y^2 + z^2}$, by distance formula.

10.4.5 : Product of vectors:

Earlier in this unit we studied the operations of vector additions, subtraction and scalar multiplication. Product of vectors is one more important concept we need to understand. There are two types of products of vectors, which are dot product and cross product.

1. Dot product of vectors: If $\bar{a}$ and $\bar{b}$ are any two vectors and θ, is angle between them (for $0 \le \theta \le \pi$), then their dot product denoted by $\bar{a} \bullet \bar{b}$ is defined as, $\bar{a} \bullet \bar{b} = |\bar{a}|\,|\bar{b}| \cos\theta$.

Using the different properties of dot product which are given below, we can verify that, if two position vectors in space are $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\bar{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then $\bar{a} \bullet \bar{b} = a_1b_1\hat{i} + a_2b_2\hat{j} + a_3b_3\hat{k}$.

Properties of dot product:

(i) For any two vectors $\bar{a}$ and $\bar{b}$ their dot product $\bar{a} \bullet \bar{b}$ is a scalar i.e. a real number, hence this product is also referred to as scalar product of vectors.

(ii) Dot product of vectors is commutative i.e. for any two vectors $\bar{a}$ and $\bar{b}$, we have $\bar{a} \bullet \bar{b} = |\bar{a}|\,|\bar{b}| \cos\theta = |\bar{b}|\,|\bar{a}| \cos\theta = \bar{b} \bullet \bar{a}$.

(iii) Any two nonzero vectors $\bar{a}$ and $\bar{b}$ are perpendicular to each other i.e. $\theta = \pi/2$ if and only if $\bar{a} \bullet \bar{b} = 0$.

(iv) If $\bar{a}$ and $\bar{b}$ are two collinear vectors then in this case $\theta = 0$ or π.
$\therefore\ \bar{a} \bullet \bar{b} = |\bar{a}|\,|\bar{b}|$ or $\bar{a} \bullet \bar{b} = -|\bar{a}|\,|\bar{b}|$ because $\cos 0 = 1$ and $\cos\pi = -1$.

(v) $\bar{a} \bullet \bar{a} = |\bar{a}|\,|\bar{a}| \cos 0 = |\bar{a}|\,|\bar{a}| = |\bar{a}|^2$.

(vi) If unit vectors along x, y and z axis are denoted by $\hat{i}, \hat{j}$ *and* $\hat{k}$ respectively. Then these are perpendicular to each other and their magnitudes are 1unit.
$\therefore\ \hat{i} \bullet \hat{j} = \hat{j} \bullet \hat{k} = \hat{k} \bullet \hat{i} = 1.1.\cos(\pi/2) = 1 \times 1 \times 0 = 0$

and

$$\hat{i} \bullet \hat{i} = \hat{j} \bullet \hat{j} = \hat{k} \bullet \hat{k} = 1.\,1.\cos 0 = 1 \times 1 \times 1 = 1.$$

(vii) Dot product is distributive over vector addition i.e.

$$\bar{a} \bullet (\bar{b} + \bar{c}) = \bar{a} \bullet \bar{b} + \bar{a} \bullet \bar{c}.$$

2. Cross product of vectors: If $\bar{a}$ and $\bar{b}$ are any two vectors and θ, is angle between them (for $0 \le \theta \le \pi$), then their cross product denoted by $\bar{a} \times \bar{b}$ is a vector whose magnitude is $|\bar{a}|\,|\bar{b}| \sin\theta$ and which has the direction perpendicular to both $\bar{a}$ and $\bar{b}$. If $\hat{n}$ denotes a unit vector perpendicular to both $\bar{a}$ and $\bar{b}$, it is such that $\bar{a}$, $\bar{b}$ and $\hat{n}$ form a right hand triplet. Then $\bar{a} \times \bar{b}$ is along $\hat{n}$. Hence, $\bar{a} \times \bar{b} = |\bar{a}|\,|\bar{b}| \sin\theta\, \hat{n}$.

If two position vectors in space are $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\bar{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then their cross product can be obtained using determinant as, $\bar{a} \times \bar{b} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times (b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \hat{i}\,(a_2 b_3 - b_2 a_3) - \hat{j}\,(a_1 b_3 - b_1 a_3) + \hat{k}\,(a_1 b_2 - b_1 a_2).$$

Using determinant it becomes easy to solve problems related with cross product.

<u>Properties of cross product:</u>

(i) For any two vectors $\bar{a}$ and $\bar{b}$ their cross product $\bar{a} \times \bar{b}$ is a vector hence this product is also referred to as vector product of vectors.

(ii) Cross product of vectors is not commutative. For any two vectors $\bar{a}$ and $\bar{b}$, we know the product $\bar{a} \times \bar{b}$ is along $\hat{n}$ but $\bar{b} \times \bar{a}$ is along $-\hat{n}$.

$\therefore\ \bar{a} \times \bar{b} = |\bar{a}|\,|\bar{b}| \sin\theta\, \hat{n}$ and $\bar{b} \times \bar{a} = |\bar{b}|\,|\bar{a}| \sin\theta\,(-\hat{n})$.

$\therefore\ \bar{a} \times \bar{b} \neq \bar{b} \times \bar{a}$.

But $\hat{n}$ is unit vector, hence $|\hat{n}| = |-\hat{n}|$, $\therefore |\bar{a} \times \bar{b}| = |\bar{b} \times \bar{a}|$.

(iii) If $\bar{a}$ and $\bar{b}$ are any two vectors which are perpendicular to each other i.e. $\theta = \pi/2$ then $\bar{a} \times \bar{b} = |\bar{a}|\,|\bar{b}|\,\hat{n}$; because $\sin \pi/2 = 1$.

(iv) If $\bar{a}$ and $\bar{b}$ are two collinear vectors then in this case $\theta = 0$ or π.

$\therefore$ Any two nonzero vectors $\bar{a}$ and $\bar{b}$ are collinear vectors if and only if $\bar{a} \times \bar{b} = 0$, because $\sin 0 = \sin\pi = 0$.

(v) $\bar{a} \times \bar{a} = |\bar{a}|\,|\bar{a}| \sin 0\, \hat{n} = 0$.

(vi) If unit vectors along x, y and z axis are denoted by $\hat{i}, \hat{j}$ *and* $\hat{k}$ respectively. Then these are perpendicular to each other and their magnitudes are 1unit.

Here $\hat{i} \times \hat{j}$ is a unit vector perpendicular to both $\hat{i}$ *and* $\hat{j}$, it is such that $\hat{i}, \hat{j}$ *and* $\hat{i} \times \hat{j}$ form a right hand triplet.

$\therefore\ \hat{i} \times \hat{j} = \hat{k}$ and $\hat{j} \times \hat{i} = -\hat{k}$,

$\hat{j} \times \hat{k} = \hat{i}$ and $\hat{k} \times \hat{j} = -\hat{i}$; similarly $\hat{k} \times \hat{i} = \hat{j}$ and $\hat{i} \times \hat{k} = -\hat{j}$.

(vii) Cross product is distributive over vector addition i.e.

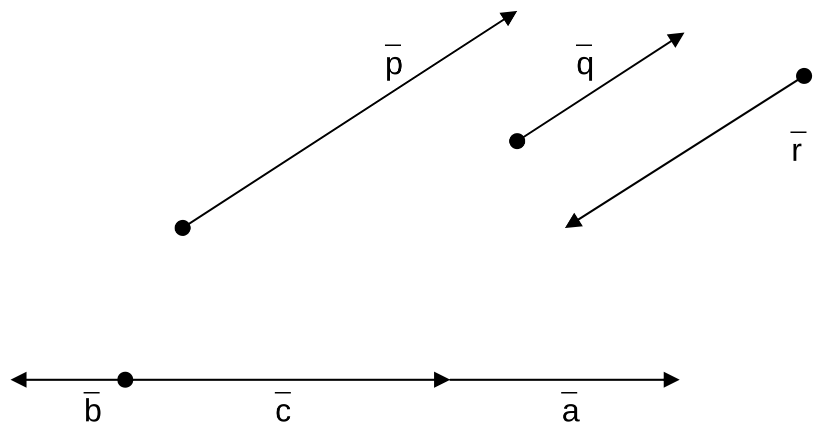
$$\bar{a} \times (\bar{b} + \bar{c}) = \bar{a} \times \bar{b} + \bar{a} \times \bar{c}.$$

10.5 : Collinear and coplanar vectors:

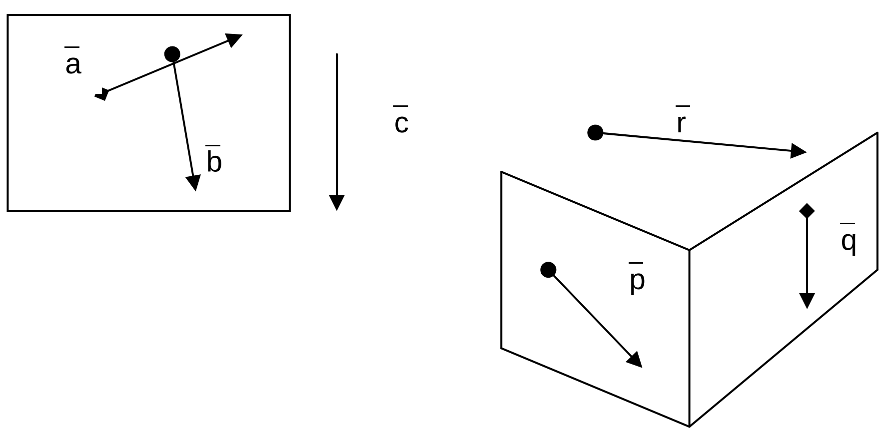
Earlier in this unit we studied the definitions of collinear and coplanar vectors. We will get some more information of these types of vectors.

<u>Collinear vectors:</u> If two or more vectors are parallel to the same line or are along the same line are called collinear vectors. The collinear vectors need not

lie on the same line always. In the following diagram vectors $\bar{p}$, $\bar{q}$ and $\bar{r}$ are collinear vectors and also vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ are collinear vectors.



<u>Coplanar vectors:</u> If more than two vectors are parallel to the same plane or are on the same plane are called coplanar vectors. The coplanar vectors need not lie on the same plane always. In the following diagram vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ are coplanar vectors as they are parallel to the same plane. But the vectors $\bar{p}$, $\bar{q}$ and $\bar{r}$ are not coplanar vectors.



<u>Some important properties of collinear and coplanar vectors:</u>

1. Two collinear vectors are scalar multiples of each other and conversely the vectors which are scalar multiples of each other are always collinear.
2. Two vectors $\bar{a}$ and $\bar{b}$ are collinear vectors if and only if $\bar{a} \times \bar{b} = 0$

3. Two nonzero vectors $\bar{a}$ and $\bar{b}$ are collinear vectors if and only if there exist two nonzero real numbers k_1 and k_2 such that $k_1 \bar{a} + k_2 \bar{b} = \bar{0}$.
4. Any two parallel or intersecting vectors form a plane hence they are always coplanar.
5. Three vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ are coplanar vectors if and only if there scalar triple product is equal to 0. i.e. $\bar{a} \bullet \bar{b} \times \bar{c} = 0$.
6. Three nonzero vectors a, $\bar{b}$ and $\bar{c}$ are coplanar if and only if there exist real numbers k_1 , k_2 and k_3 not all zero such that $k_1 \bar{a} + k_2 \bar{b} + k_3 \bar{c} = \bar{0}$.

***Self Test II*:**

10.6: Summary for Unit 10:

In this unit learners studied the following topics in details:

1. The concept of vector quantities and scalar quantities.
2. Different types of vectors such as, equal vectors, unit vector, zero vector, collinear vectors and coplanar vectors etc.
3. Vector operations such addition, subtraction and scalar multiplication, dot product of vectors and cross product of vectors.
4. Representation of a point in plane or in space using vectors.