Unit 9: Functions

9.0 Unit Objectives:

By the end of this Unit, learners should be able to:

- Understand concept of function.
- Describe Injective or one to one function.
- Describe surjective or onto function.
- Describe bijective function.
- Find inverse function of a bijective function.
- Find composition of functions.

9.1 : Unit Introduction:

The concept of function is of fundamental importance in mathematics. Function is a special type of relations which plays an important role in mathematics, physics, computer science and many other fields.

Functions are used in a variety of situations. While studying Geometry in school, we learn that area of a circle of radius r units equals to πr^2 square units. We say that area of a circle is a function of radius r of that circle. In physics we learn about velocity of a vehicle at time t, velocity is a function of time t. Functions can be of one, or two or more variables. In this unit we study functions of one variable, its basic properties and different types.

9.2 : Functions

Definition 9.2.1 : Function: If A and B are nonempty sets then , a function f from A to B denoted by $f : A \to B$ is a relation from A to B such that every element of set A is related to a unique element of set B. Function is also referred to as a mapping.

If $x \in A$ is related to $y \in B$ by the function f i. e. $(x, y) \in f$, then we express it as $y = f(x)$. In this case y is called as a image of x and x is called as a preimage of y.

If $f : A \to B$ is a function then A is called as domain of the function f and B is called as co-domain of the function f. And the set of images which is a subset of b is called as the range of the function f.

Examples:

- If A = {3, 5, 8, 15} and B= {5, 10, 15}.

Let f = { (3 ,5), (5,5), (8,15), (15 ,15)}, then f is a relation from set A to set B as $f \subset A \times B$. And relation f is a function from set A to Set B, because every element of set A is related to a unique element of set B. i.e. here $f : A \to B$ is a function and we can define it as f(3) = 5, f(5) = 5, f(8) = 15 and f(15) = 15.

Note that two elements 3 and 5 of set A, are having the image 5, but for each of them it is unique image. Also the two elements 8 and 15 of set A, are having the same image 15, but for each of them it is unique image. So f is a function.

For this function f, the domain of f = {3, 5, 8, 15} ,
the co -domain of f = { 5, 10, 15} and the range of f = {5, 15}.

- If A = {3, 5, 8, 15} and B= {5, 10, 15}.

Let g = {(3 ,5), (3,10), (5,10), (8,15), (15,15) }, then g is a relation from set A to set B as g $\subset$ A $\times$ B. But the relation g is not a function from set A to Set B, because every element of set A is not related to a unique element of set B. We observe that an element 3 of set A is having two images 5 and 10, and hence by the definition of function g is not a function.

- If A = {3, 5, 8, 15} and B= {5, 10, 15} and k = {(3 ,5), (8,10), (15,15) }, then k is a relation from set A to set B as k $\subset$ A $\times$ B. But the relation k is not a function from set A to Set B, because every element of set A is not related to an element of set B. We observe that an element 5 of set A, is not related to any of the elements from set B, and hence by the definition of function k is not a function.

- On set A = {1, 3, 5, 7} define a relation f as, for a$\in$A and b$\in$B, (a, b) $\in$ f if a $<$ b. i.e. f(a) = b if a $<$ b.

We observe that by this relation 1 is related to 3, 5 and 7. So 1 in set A has three images but by the definition of function each element in the domain set must have a unique image.

Also 7 in A is not related to any of the other elements and by the definition of function each element in the domain set must have image which must be unique. Hence this relation f is not a function.

- Let A = { –2, –1, 1, 2, 3 } and B = { 1, 4, 9, 25 }. Define a relation g from set A to set B as (a, b) $\in$R i.e. g(a) = b if and only if b = a^2.

Then g as a set of ordered pairs can be written as

g = { (–2, 4), (–1,1), (1,1), (2, 4), (3, 9) } We observe that –1 and 1 from A are related to unique element 1, –2 and 2 from A are related to unique element 4 and also 3 is related to unique element 9 of set B. So this relation satisfies the definition of a function. In this case we write that ”
$f : A \to B$ is a function defined as f(x) = x^2 “.

9.3: Types of functions:

Definition 9.3.1 : Injective(or one to one) function: A function $f : A \rightarrow B$ is said to be an injective function if distinct elements of A have distinct images in B under f. An injective function is also called as a one to one function.

Note that: The function $f : A \rightarrow B$ is an injective function if for x, y$\in$A, f(x) = f(y) implies that x = y.

Examples:

- Let A = {1, 3, 5, 7, 9} and B = {1, 9, 25, 49, 81, 100}. Define a function $f : A \rightarrow B$ as f(x) = x^2.

  Then f(1) = 1, f(3) = 9, f(5) = 25, f(7) = 49 and f(9) = 81.

  We observe that every element of A is related to a different element of set B, so this function is injective function from set A to set B.

- Let A = {1, 3, 5, 7} and B = {1, 2, 3, **…**, 10 }. Define a function $f : A \rightarrow B$ as f(x) = x.

  Then f(1) = 1, f(3) = 3, f(5) = 5, 5 and f(7) = 7.

  We observe that every element of A is related to a different element of set B, so this function is injective function from set A to set B.

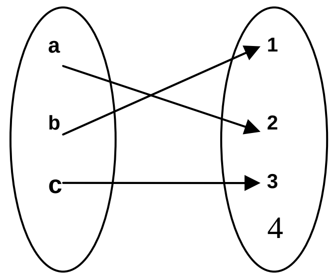
- Let A = { –2, –1, 1, 2, 3 } and B = { 1, 4, 9, 25 }. Define a function $f : A \rightarrow B$ as f(x) = x^2.

  Then f(–2) = 4 , f(–1) = 1, f(1) = 1, f(2) = 4 and f(3) = 9.

We observe that every element of A is not related to a different element of set B. Here –1 and 1 from A are related to 1,so f(–1) = 1 = f(1) but it does not imply that –1 = 1.

Also f(–2) = 4 = f(2) but –2 $\neq$ 2 so this function is not injective function .

- The Venn diagram below represents an injective function from set A = {a, b, c} to the set {1, 2, 3, 4}



Injective function

Definition 9.3.2: **<u>Surjective (or onto)</u>** <u>function:</u> A function $f : A \to B$ is said to be an surjective function if every element of set B (i.e. codomain), has at least one preimage in set A (i.e. domain). A surjective function is also called as an onto function.

Examples:

- Let A = { –2, –1, 1, 2, 3 } and B = { 1, 4, 9}. Define a function $f : A \to B$ as f(x) = x^2.

  Then f(–2) = 4 , f(–1) = 1, f(1) = 1, f(2) = 4 and f(3) = 9.

We observe that every element of set B, has at least one preimage in set A. So this function is a surjective function from set A to set B.

- If A = {3, 5, 8, } and B= { 5, 10, 15}.

  Let f = { (3 ,5), (5, 15), (8,10) }, then f is a function from set A to set B such that every element of set B is related to a unique element of set A. So by definition this function is a surjective function from set A to set B.

- Let A = {1, 3, 5, 7, 9} and B = {1, 9, 25, 49, 81, 100}. Define a function $f : A \to B$ as f(x) = x^2.

  Then f(1) = 1, f(3) = 9, f(5) = 25 , f(7) = 49 and f(9) = 81.

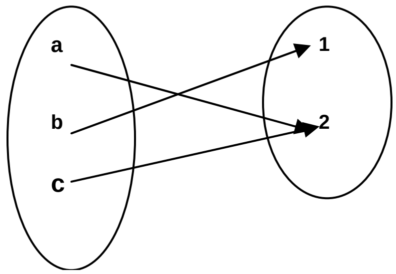
  We observe that every element of B is not related with an element of set A, in particular the number 100 in set B has no preimage in set A. So this function is not surjective function from set A to set B.

- Let A = {1, 3, 5, 7} and B = {1, 2, 3, **…**, 10 }. Define a function $f : A \to B$ as f(x) = x.

  Then f(1) = 1, f(3) = 3, f(5) = 5, 5 and f(7) = 7.

  We observe that every element of B is not related with an element of set A. So this function is not surjective function from set A to set B.

  - The Venn diagram below represents a surjective function from set A = { a, b, c} to the set { 1, 2 }



Surjective function

Definition 9.3.3 : Bijective function **:** A function $f : A \to B$ is said to be a bijective function if it is injective and surjective both. A bijective function is also called as a one to one and onto function.

Examples:

- If A = {3, 5, 8, } and B= { 5, 10, 15}.

  Let f = { (3 ,5), (5, 15), (8,10) }, then f is a function from set A to set B which is injective as all images are distinct. Also every element of set B has a unique preimage under f in set A. So by definition this function is a surjective function.

Therefore it is a bijective function from set A to set B.

- Let A = {1, 3, 5, 7, 9} and B = {1, 9, 25, 49, 81}. Define a function $f : A \to B$ as $f(x) = x^2$.

  Then f(1) = 1, f(3) = 9, f(5) = 25, f(7) = 49 and f(9) = 81.

  As discussed above this function is injective function from set A to set B. and it is surjective function as every element of set B is an image of some element from set A. Therefore it is a bijective function.

- Let A = {1, 3, 5, 7} and B = {1, 2, 3, **…**, 10 }. Define a function $f : A \to B$ as f(x) = x.

  Then f(1) = 1, f(3) = 3, f(5) = 5, 5 and f(7) = 7.

  As discussed above this function is injective function from set A to set B. But it is not surjective function, because the numbers 2, 4, 6, 8, 9, 10 from set B have no preimages in set A under the function f. Therefore it is not a bijective function.

- Let A = { –2, –1, 1, 2, 3 } and B = { 1, 4, 9, 25 }. Define a function $f : A \to B$ as $f(x) = x^2$.

  Then f(–2) = 4 , f(–1) = 1, f(1) = 1, f(2) = 4 and f(3) = 9.

  As discussed above this function is not injective function from set A to set B, because f(–1) = 1 = f(1) but $-1 \neq 1$ and f(–2) = 4 = f(2) but $-2 \neq 2$.

Therefore it can not be a bijective function. But observe that it is a surjective function, as all the numbers from set B have preimages in set A under the function f.

Note that: for a function $f : A \to B$, being injective (or surjective) function does not depend only on the definition of the function or only on the sets A and B. So if a function $f : A \to B$ is injective (or surjective) function from set A to set B then the same function(i.e. rule), may not be injective (or surjective) function from some other set C to some other set D.

e.g. from above examples $f : A \to B$ as $f(x) = x^2$ is a bijective function for one pair of sets A and B while it is not a bijective function for another pair of sets A and B.

- If $f : \Re \to \Re$ is a function defined as $f(x) = 3x + 2$, where $\Re$ is a set of real numbers then is f a bijective function? Here $\Re$ denotes the set of real numbers.

<u>Solution:</u> To determine whether this function a bijective function or not we should determine whether it is an injective function and a surjective function.

i) Injective: Let $x, y \in \Re$ are such that $f(x) = f(y)$.

$$\therefore \quad 3x + 2 = 3y + 2$$

$$\therefore \quad 3x = 3y$$

$$\therefore \quad x = y.$$

Hence for $x, y \in \Re$, we know , $f(x) = f(y)$ implies that $x = y$. So this function $f : \Re \to \Re$ is an injective function.

i) Surjective: Let $a, b \in \Re$ are such that $f(a) = b$.

$$\therefore \quad 3a + 2 = b$$

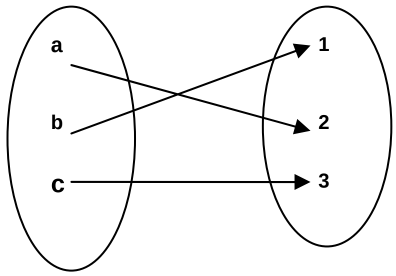
$$\therefore \quad 3a = b - 2$$

$$\therefore \quad a = \frac{b-2}{3}$$

Hence for every $b \in \Re$ (i.e in codomain), we can find preimage $a \in \Re$ (i.e in domain) . This implies that this function $f : \Re \to \Re$ is a surjective function.

From above explanation we can say that this function $f : \Re \to \Re$ is a bijective function.

- The Venn diagram below represents a bijective function from set A = {a, b, c} to the set {1, 2, 3}



Bijective function

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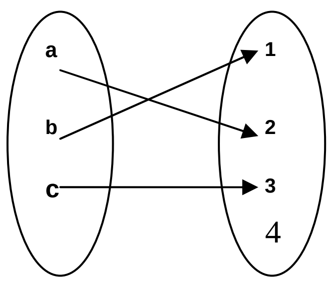
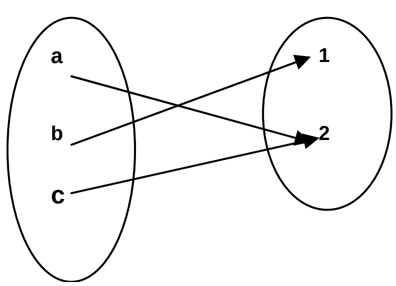
- The Venn diagram below represents an injective function from set A = {a, b, c} to the set {1, 2, 3, 4} but it is not a surjective function as 4 from set B has no preimage in A . Hence this function is not a bijective function.



Injective function

- The Venn diagram below represents a surjective function from set A = { a, b, c} to the set { 1, 2 }, but it is not an injective function as the images are not distinct. Here f (a) = 2 = f(c) but a and c are different elements from A. Hence this function is not a bijective function.

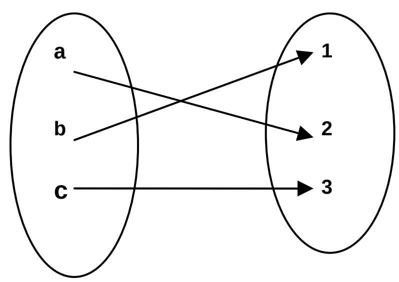


Surjective function

***Self Test I*:**

9.4: Inverse function:

Consider the bijective function represented by a Venn diagram given below. This is a function $f : A \to B$, where set A = {a, b, c} and B = {1, 2, 3}



Under this function f(a) = 2, f(b) = 1 and f(c) = 3. Hence preimage of 2 is a, preimage of 1 is b and preimage of 3 is c. If we consider the preimages under function f we obtain one function from the set B to the set A. This function is called inverse function of f.

Definition 9.3.4 : Inverse function: If $f : A \to B$ is a bijective function then its inverse function denoted by f^{-1} exists. And the inverse function $f^{-1} : B \to A$ is defined as, if f(x) = y then $f^{-1}(y) = x$.

Note that : Inverse of function f exists if and only if f is bijective function.

Examples:

- If A = {3, 5, 8, } and B= { 5, 10, 15} and f is a function from set A to set B, where as a relation f can be represented as a set of ordered pairs f = { (3 ,5), (5, 15), (8,10) }. As discussed earlier this function is a bijective function from set A to set B.

Therefore inverse function of f exists. The inverse function, denoted by f^{-1} is a function from set B to set A and it can be represented as a set of ordered pairs , $f^{-1} = \{ (5,3), (15, 5), (10,8) \}$.

- Let A = {1, 3, 5, 7, 9} and B = {1, 9, 25, 49, 81}. Define a function $f : A \to B$ as $f(x) = x^2$.

  Then f(1) = 1, f(3) = 9, f(5) = 25, f(7) = 49 and f(9) = 81.

  As discussed above this function is a bijective function. So there exists inverse function f^{-1} from set B to set A, which is such that $f^{-1}(1) = 1$, $f^{-1}(9) = 3$, $f^{-1}(25) = 5$, $f^{-1}(49) = 7$ and $f^{-1}(81) = 9$.

In this case we can describe this inverse function as, $f^{-1} : B \to A$ is defined as,

$$f^{-1}(x) = \sqrt{x}\ .$$

If $f : \Re \to \Re$ is a function defined as f(x) = 3x + 2, then this function is a bijective function as verified earlier.

For x, y $\in \Re$, we know , if f(x) = y, then

$$3x + 2 = y$$

$$\therefore \quad 3x = y - 2$$

$$\therefore \quad x = \frac{y-2}{3}$$

So the inverse function is described as , $f^{-1} : \Re \to \Re$ and $f^{-1}(y) = x = \dfrac{y-2}{3}$.

As it is common practice to write all function definitions using variable x , so this definition of inverse function is generally written as $f^{-1}(x) = \dfrac{x-2}{3}$

9.4 : Composition of functions:

While studying functions in details we need to perform different operations on functions such as addition or difference etc. Addition and difference of functions are very simple operations to understand. So here we will study the composition of functions, which is the most important operation of functions.

Definition 9.4.1 : Composition of functions: If $f : A \to B$ and $g : B \to C$ are any two functions , then the Composition of functions f and g denoted by $g \circ f$ is a function from A to C i.e. $g \circ f : A \to C$ is a function defined by ,

$g \circ f(x) = g[f(x)]$, for all $x \in A$.

Note that:

(i) The composition of two functions f and g is defined only when the codomain of the first function is identical to the domain of the second function.

(ii) The composition of functions is not a commutative operation.

(iii) The composition of functions is an associative operation.

(iv) The composition $f \circ f$ (if it exists) is also denoted by f^2.

Examples:

- Let A = {1, 2, 3, } , B = { 5, 15, 25, 35 } and C = { 7 , 17, 27,37 }. Let $f : A \to B$ be defined as f(1) = 15, f(2) = 25 , f(3) = 35 and $g : B \to C$ be defined as g(5) = 7, g(15) = 17 , g(25) = 27 and g(35) = 37.

  Now the Composition of functions f and g denoted by $g \circ f$ is a function, $g \circ f : A \to C$ defined by , $g \circ f(x) = g(f(x))$, for all $x \in A$.

$\therefore$ $g \circ f(1) = g(f(1)) = g(15) = 17$; $g \circ f(2) = g(f(2)) = g(25) = 27$ and $g \circ f(3) = g(f(3)) = g(35) = 37$.

Observe that in this case the composition $f \circ g$ does not exist.

- Let A = { –1, –2, –3 } , B = { 1, 4, 9, 10 } and C = {1,2,3,…,15 } are three sets. Define a function $f : A \to B$ as $f(x) = x^2$ and another function $g : B \to C$ as g(x) = x + 3. Then f(–1) = 1, f(–2) = 4 and f(–3) = 9 . Also g(1) = 4, g(4) = 7 , g(9) = 12 and g(10) = 13.

Now the Composition, $g \circ f : A \to C$ is a function defined by , $g \circ f(x) = g(f(x))$, for all $x \in A$.

$\therefore$ $g \circ f(-1) = g(f(-1)) = g(1) = 4$; $g \circ f(-2) = g(f(-2)) = g(4) = 7$ and $g \circ f(-3) = g(f(-3)) = g(9) = 12$.

Observe that in this case the composition $f \circ g$ does not exist.

- Let N denotes the set of natural numbers. Define functions $f : \mathrm{N} \to \mathrm{N}$ and $g : \mathrm{N} \to \mathrm{N}$ as f(x) = 2x + 1 and $g(x) = x^2$. Then Composition, $g \circ f : \mathrm{N} \to \mathrm{N}$ is a function defined by, $g \circ f(x) = g(f(x))$, for all $x \in A$.

$\therefore$ $g \circ f(x) = g(f(x)) = g(2x + 1) = (2x + 1)^2 = 4x^2 + 4x + 1$.

Observe that in this case the composition $f \circ g$ also exists. And the composition, $f \circ g : \mathrm{N} \to \mathrm{N}$ is a function defined by, $f \circ g(x) = f(g(x))$, for all $x \in A$.

$\therefore$ $f \circ g(x) = f(g(x)) = f(x^2) = 2x^2 + 1$.

Obviously $g \circ f(x) \neq f \circ g(x)$. $\therefore$ The composition of functions is not a commutative operation.

Further note that in this case the compositions f o f and g o g also exist.
The composition $f \circ f : \mathrm{N} \to \mathrm{N}$ is defined as

$$f \circ f\,(x) = f(\,f\,(\,x)\,) = f\,(2x + 1) = 2\,(2x + 1) + 1 = 4x + 3.$$

The composition $g \circ g : \mathrm{N} \to \mathrm{N}$ is defined as

$$g \circ g\,(x) = g(\,g\,(\,x)\,) = g\,(x^2) = (\,x^2\,)^2 = x^4.$$

- Let $\mathbb{R}$ denotes the set of natural numbers. Define functions $f : \mathrm{R} \to \mathrm{R}$ and $g : \mathrm{R} \to \mathrm{R}$ as $f(x) = x^2 + x$ and $g(x) = 5x + 7$.

Then Composition, $g \circ f : R \to R$ is a function defined by,

$$g \circ f\,(x) = g(\,f(\,x)\,) = g\,(\,x^2 + x\,) = 5\,(\,x^2 + x\,) + 7 = 5\,x^2 + 5x + 7.$$

And the composition, $f \circ g : R \to R$ is a function defined by,

$$f \circ g\,(x) = f(\,g\,(\,x)\,) = f\,(5x + 7) = (5x + 7)^2 + (5x + 7)$$
$$= 25\,x^2 + 70\,x + 49 + 5x + 7 = 25\,x^2 + 75\,x + 56.$$

The composition $f \circ f : R \to R$ is defined as

$$f \circ f\,(x) = f(\,f\,(\,x)\,) = f\,(x^2 + x\,) = (x^2 + x\,)^2 + (x^2 + x\,) = (\,x^2)^2 + 2x^3 + x^2 + (x^2 + x\,)$$
$$= x^4 + 2x^3 + 2x^2 + x\,.$$

The composition $g \circ g : R \to R$ is defined as

$$g \circ g\,(x) = g(\,g\,(\,x)\,) = g\,(5x + 7) = 5(5x + 7) + 7 = 25\,x + 35 + 7 = 25\,x + 42\,.$$

***Self Test II*:**

9.5: Summary for Unit 9:

In this unit learners studied the following topics in details:

1. The concept of function which is a special type of relation.
2. Definition and examples of Injective or one to one function
3. Definition and examples of surjective or onto function.
4. Definition and examples of Bijective function.
5. Inverse function of a bijective function
6. The concept of the composition of functions.