# Unit 14: Polynomials

## 14.0 Unit Objectives:

By the end of this Unit, learners should be able to:

- Define polynomials in one variable
- Find the degree of polynomial
- Understand Equality of Polynomials
- Perform addition, subtraction , multiplication and division of polynomials
- Determine factors and roots of polynomial equation
- Quadratic equations and to find the roots of Quadratic equations

## 14.1 Unit Introduction:

Polynomials are one of the most important concepts in algebra and throughout mathematics, science and engineering. They are used to form polynomial equations which are used to solve a wide range of problems which appear in basic chemistry, physics and even economics.. Polynomials are also used to approximate other functions. Polynomials are widely used because they are flexible and can take many forms of data. The generating functions are special type of polynomials which are used as a powerful technique for solving complicated problems from combinatorics an important branch of applied mathematics. In this unit we will study about polynomials in one variable and the related terminology. We will also study about different operations on polynomials and about the roots of polynomials.

## 14.2 : Polynomials :

In mathematics a polynomial is an expression constructed from variables and constants using the operations of addition, subtraction, multiplication and raising to constant non negative powers. For example $5x^2+9x+4$ is a polynomial but $6x^2+9x+4x^{1/2}$ is not a polynomial because the later involves an exponent which is not an integer. We will start the discussion with the formal definition of polynomial.

**Definition 14.2.1 :** <u>Polynomial</u>**:**

An algebraic expression of the form $a_0+ a_1x + a_2x^2 + \ldots + a_nx^n$ is called a "polynomial in variable x over real numbers", where n is either 0 or a positive integer and $a_0$, $a_1$, $a_2$, **…** , $a_n$ are real numbers which are called coefficients.

Polynomials are generally denoted as function of x i.e. as f(x), p(x), q(x) etc.

**Definition 14.2.2** <u>: Degree of Polynomial</u>**:** The degree of the polynomial is the exponent in the term with the highest power.

If $p(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$ is a polynomial such that $a_n \neq 0$, then p(x) is a polynomial of degree n.

A polynomial of degree 1 is also called a linear polynomial, a polynomial of degree 2 is called a quadratic polynomial, and a polynomial of degree 3 is called a cubic polynomial.

**Definition 14.2.3** <u>:</u> <u>Constant polynomial:</u>

If $p(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$ is a polynomial such that $a_0 \neq 0$ and $a_1 = 0$, $a_2 = 0, \dots, a_n = 0$, then $p(x) = a_0$ and it is called a constant polynomial. Clearly, a constant polynomial is of degree 0. As the highest power term is $x^0$.

**Definition 14.2.4** <u>:</u> <u>Zero polynomial:</u>

If $p(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$ is a polynomial such that all coefficients are 0 i.e. $a_0 = 0$, $a_1 = 0, \dots, a_n = 0$, then p(x) is called a zero polynomial. For a zero polynomial the degree is not defined.

Examples:

- $f(x) = 3x - 2$, is a polynomial of degree 1, in this polynomial coefficient of x is 3 and the coefficient of $x^0$ or the constant term is –2 .
- $g(x) = 9x^2 - \frac{2}{x}$ , it is not a polynomial. (Why?)
- $p(x) = -6x^2 + 9x + \frac{1}{2}$ , is a polynomial of degree 2 i.e. a quadratic polynomial. In this polynomial coefficient of $x^2$ is – 6 , coefficient of x is 9 and the constant term is $\frac{1}{2}$ .
- $q(x) = 4x^3 + \frac{1}{4}x - 4$, is a polynomial of degree 3 i.e. a cubic polynomial. In this polynomial coefficient of $x^3$ is 4, coefficient of $x^2$ is 0 , coefficient of x is $\frac{1}{4}$ and the constant term is – 4 .
- $r(x) = 5x^7$ is a polynomial of degree 7.. In this polynomial coefficient of $x^7$ is 5 and all other coefficients are zero.
- $g(x) = 8$ , is a constant polynomial as it is of degree 0.
- $z(x) = 0 = 0x + 0 = 0x^3 + 0x^2 + 0x + 0$, is a zero polynomial.

## 14.3 : Equality of Polynomials

**Definition 14.3.1 :** Equal polynomials:

Two polynomials are said two be equal if the coefficients of like powers of x are equal. i.e. the polynomials $p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ and $q(x) = b_0 + b_1x + b_2x^2 + \ldots + b_nx^n$ are equal if $a_0 = b_0$, $a_1 = b_1$, $a_2 = b_2$, …, $a_n = b_n$.

## 14.4 : Operations on polynomials:

14. 4 .1: Multiplication of a polynomial by a real number : If k is a real number and $p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ is any polynomial then $k.p(x)$ is the polynomial in which the coefficients are $k^{th}$ multiples of the coefficients of p(x ). i.e. $k\,p(x) = ka_0 + ka_1x + ka_2x^2 + \ldots + ka_nx^n$.

Examples:

- $f(x) = 3x + 2$, $\therefore -2f(x) = -2(3x+2) = -6x - 4$.
- $p(x) = 6x^2 + 9x + 2$, $\therefore \frac{1}{3}\, f(x) = \frac{1}{3}(6x^2 + 9x + 2) = 2x^2 + 3x + \frac{2}{3}$.

14. 4 .2: Addition of two polynomials:

The addition of two polynomials is obtained by adding the terms of like powers.

If $p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ and $q(x) = b_0 + b_1x + b_2x^2 + \ldots + b_mx^m$ are two polynomials and m is greater than n then,

$$p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \ldots + (a_n + b_n)x^n + b_{n+1}x^{n+1} + \ldots + b_mx^m \quad b_mx^m$$

If p(x) is a polynomial of degree n and q(x) is a polynomial of degree m, then the degree of the addition polynomial p(x) + q(x) is equal to the greater of the two integers m and n.

Examples:

- $f(x) = 3x - 2$ and $g(x) = 8x^2 + 9x + 3$ are two polynomials , then their addition is $f(x) + g(x) = (3x - 2) + (8x^2 + 9x + 3) = (0+8)x^2 + (3+9)x + (-2+3)$
$$= 8x^2 + 12x + 1.$$
- $p(x) = 6x^3 + 9x^2 + \frac{1}{2}$ and $q(x) = 4x^3 + \frac{1}{4}x - 4$, are two polynomials , then their addition is $p(x) + q(x) = (6x^3 + 9x^2 + \frac{1}{2}) + (4x^3 + \frac{1}{4}x - 4)$
$$= (6+4)x^3 + (9+0)x^2 + (0+\tfrac{1}{4})x + (\tfrac{1}{2} - 4)$$
$$= 10x^3 + 9x^2 + \tfrac{1}{4}x - \tfrac{7}{2}.$$

## 14. 4 .3: Difference of two polynomials:

The difference of two polynomials is obtained by subtracting the terms of like powers.

If $p(x) = a_0 + a_1x + a_2 x^2 + \ldots + a_n x^n$ and $q(x) = b_0 + b_1x + b_2 x^2 + \ldots + b_m x^m$ are two polynomials and m is greater than n then,

$$p(x) - q(x) = (a_0 - b_0) + (a_1 - b_1) x + (a_2 - b_2) x^2 + \ldots + (a_n - b_n)x^n - b_{n+1}x^{n+1} - \ldots - b_m x^m \quad b_m x^m .$$

If p(x) is a polynomial of degree n and q(x) is a polynomial of degree m, then the degree of the difference polynomial $p(x) - q(x)$ is equal to the greater of the two integers m and n.

Examples:

- $f(x) = 3x - 2$ and $g(x) = 8x^2 + 9x + 3$ are two polynomials , then their difference is $f(x) - g(x) = (0x^2 + 3x - 2) - (8x^2 + 9x + 3)$

$$= (0 - 8) x^2 + (3 - 9) x + (-2 - 3)$$
$$= -8x^2 - 6x - 5.$$

- $p(x) = 6x^3 + 9x^2 + \frac{1}{2}$ and $q(x) = 4x^3 + \frac{1}{4} x - 4$, are two polynomials , then their difference is $p(x) - q(x) = (6x^3 + 9x^2 + \frac{1}{2}) - (4x^3 + \frac{1}{4} x - 4)$

$$= (6 - 4)x^3 + (9 - 0)x^2 + (0 - \tfrac{1}{4}) x + (\tfrac{1}{2} - (-4))$$
$$= 2x^3 + 9x^2 - \tfrac{1}{4}x + \tfrac{9}{2}.$$

## 14. 4 .4: Multiplication of two polynomials:

If $p(x) = a_0 + a_1x + a_2 x^2 + \ldots + a_n x^n$ and $q(x) = b_0 + b_1x + b_2 x^2 + \ldots + b_m x^m$ are two polynomials having degrees n and m respectively then, their multiplication is the polynomial, obtained by term by term multiplication.

$$\therefore\ p(x) \times q(x) = c_0 + c_1x + c_2 x^2 + \ldots + c_k x^k, \text{ where } c_k = \sum_{i=0}^{k} a_i \times b_{k-i} .$$

i.e. $c_0 = (a_0 \times b_0)$, $c_1 = (a_1 \times b_0 + a_0 \times b_1)x$ and $c_2 = (a_2 \times b_0 + a_1 \times b_1 + a_0 \times b_2)$ etc.

If p(x) is a polynomial of degree n and q(x) is a polynomial of degree m, then the degree of the multiplication polynomial $p(x) \times q(x)$ is equal to $m \times n$.

Examples:

- $f(x) = 3x - 2$ and $g(x) = 8x^2 + 9x + 3$ are two polynomials , then their product or multiplication is $f(x) \times g(x) = (3x - 2) \times (8x^2 + 9x + 3)$

$$= 3x \times (8x^2 + 9x + 3) - 2 \times (8x^2 + 9x + 3)$$
$$= 24x^3 + 27x^2 + 9x - 16x^2 - 18x - 6$$

$$= 24\,x^3 + 11\,x^2 - 9x\ -6$$

- $p(x) = 6\,x^3 + 9x^2 + ½$ and $q(x) = 4x^3 + ¼\,x - 4$, are two polynomials ,

then their multiplication is $p(x) \times q(x) = (6\,x^3 + 9x^2 + ½\,) \times (4x^3 + ¼\,x - 4)$

$$= 6\,x^3 \times (4x^3 + ¼\,x - 4) + 9x^2 \times (4x^3 + ¼\,x - 4) + ½ \times (4x^3 + ¼\,x - 4)$$

$$= (24\,x^6 + \tfrac{6}{4}\,x^4 - 24x^3\,) + (36\,x^5 + \tfrac{9}{4}\,x^3 - 36\,x^2\,) + (\,2x^3 + \tfrac{1}{8}\,x - 2)$$

$$= 24\,x^6 + 36\,x^5 + \tfrac{6}{4}\,x^4 - \tfrac{79}{4}\,x^3 - 36\,x^2 + \tfrac{1}{8}\,x - 2\ .$$

## 14. 4 .5: <u>Division of two polynomials:</u>

Let $p(x) = a_0 + a_1x + a_2\,x^2 + \ldots + a_n\,x^n$ and $q(x) = b_0 + b_1x + b_2\,x^2 + \ldots + b_m\,x^m$ are two polynomials having degrees n and m respectively with $m \le n$. If there exists a polynomial t(x) such that, $p(x) = t(x) \times q(x)$ ,then we say that “p(x) is divisible by q(x)” or “q(x) is a factor of p(x).” The division or the quotient t(x) is a polynomial of degree m – m and it is obtained by step by step procedure explained below.

Examples:

- $f(x) = 3x - 2$ and $g(x) = 6\,x^2 + 9x + 3$ are two polynomials , then their

division $g(x) \,/\, f(x) = (6\,x^2 + 9x + 3\,) \,/\, (3x - 2)$ is obtained as follows.

First the highest degree term in g (x) is divided by the highest degree term in f(x), and by that division every term in g(x) is multiplied. This product is the subtracted from g (x). Then highest degree term in the subtraction obtained is divided by the highest degree term in f(x) and the above procedure is repeated again and again till it is possible.

$$
\begin{array}{l|l}
 & 2x\ + 4 \\
\hline
3x - 2 & 6\,x^2\ + 8x - 8 \\
 & 6\,x^2\ - 4x \\
 & -\qquad + \\
\hline
 & \qquad 12x - 8 \\
 & \qquad 12x - 8 \\
 & \qquad -\quad + \\
\hline
 & \qquad 0x + 0
\end{array}
$$

$$\therefore \frac{6x^2 + 8x - 8}{3x - 2} = 2x + 4$$

- The division $(x^3 - 6x^2 + 11x - 6) / (x - 2)$ is obtained as follows.

| | |
|---|---|
| | $x^2 - 4x + 3$ |
| $x - 2$ | $x^3 - 6x^2 + 11x - 6$ |
| | $x^3 - 2x^2$ |
| | $-\quad +$ |
| | $-4x^2 + 11x - 6$ |
| | $-4x^2 + 8x$ |
| | $+\quad -$ |
| | $3x - 6$ |
| | $3x - 6$ |
| | $-\quad +$ |
| | $0$ |

$$\therefore \frac{x^3 - 6x^2 + 11x - 6}{x - 2} = x^2 - 4x + 3$$

- $p(x) = 24x^6 + 36x^5 + \frac{6}{4}x^4 - \frac{79}{4}x^3 - 36x^2 + \frac{1}{8}x - 2$

and $q(x) = 4x^3 + \frac{1}{4}x - 4$, are two polynomials, then their division $p(x)/q(x)$ is obtained as follows.

| | |
|---|---|
| | $6x^3 + 9x^2 + \frac{1}{2}$ |
| $4x^3 + \frac{1}{4}x - 4$ | $24x^6 + 36x^5 + \frac{6}{4}x^4 - \frac{79}{4}x^3 - 36x^2 + \frac{1}{8}x - 2$ |
| | $24x^6 + 0 + \frac{6}{4}x^4 - 24x^3$ |
| | $-\quad +\quad -\quad +$ |
| | $36x^5 + 0 + \frac{17}{4}x^3 - 36x^2$ |

| | |
|---|---|
| | $36x^5 \quad +0 \quad + \frac{9}{4}x^3 - 36x^2$ |
| | $+ \quad\quad - \quad\quad +$ |
| | $2x^3 \quad +0 \quad + \frac{1}{8}x - 2$ |
| | $2x^3 \quad +0 \quad + \frac{1}{8}x - 2$ |
| | $- \quad\quad - \quad\quad +$ |
| | 0 |

$\therefore \frac{p(x)}{q(x)} = 6x^3 + 9x^2 + ½$ .

***Self Test I*:**

14. 4 .6: <u>Synthetic division</u> : If for polynomials p(x) and q(x), there exists a polynomial t(x) such that, p(x) = t(x) × q(x) ,then we say that "q(x) is a factor of p(x)." If this factor q(x) is linear i.e. if the polynomial q(x) is of degree one and has the form q(x) = x – a, then the polynomial t(x) can be found using synthetic division which is easier that usual polynomial division.

This method is explained with examples below:

Examples:

- Find $(x^3 - 6x^2 + 11x - 6)$ / $(x - 2)$ using the synthetic division.

<u>Solution:</u> Here $p(x) = x^3 - 6x^2 + 11x - 6$ and $q(x) = x - 2$ . We will write a table in which the first row , second column contains the coefficients 1, –6, 11 and –6 of p(x) in the decreasing order of powers of x. We also write +2 in the first column of the table as the factor of p(x) is x – 2. Write the first coefficient 1 in third row of the table. Multiply the coefficients –6, 11 and –6 by 2 and write each product below each of the coefficients. In the next row perform the addition of the numbers previously written. This addition gives the coefficients of the required quotient polynomial. The remainder is written in a rectangle. If the remainder is not zero then q(x) is not a factor of p(x).

| | | | | |
|---|---|---|---|---|
| | 1 | –6 | 11 | –6 |
| 2 | ↓ | 2 | – 8 | 6 |
| | 1 | –4 | 3 | $\boxed{0}$ |

From the third row the coefficients of the required quotient polynomial are 1, –4 and 3. So the quotient is the polynomial $x^2 - 4x + 3$. And the factorization of p(x) is, $p(x) = x^3 - 6x^2 + 11x - 6 = (x - 2) \times (x^2 - 4x + 3)$.

- Find $(4x^3 - 20x^2 + 17x - 4) / (x - 4)$ using the synthetic division.

Solution: Here $p(x) = 4x^3 - 20x^2 + 17x - 4$ and $q(x) = x - 4$. We will write a table in which the first row, second column contains the coefficients 4, –20, 17 and – 4 of p(x) in the decreasing order of powers of x. We also write +4 in the first column of the table as (x – 4) is the factor of p(x). Write the first coefficient 4 in third row of the table. Multiply the coefficients –20, 17 and – 4 of p(x) by 4 and write each product below each of these coefficients. In the next row perform the addition of the numbers previously written. This addition gives the coefficients of the required quotient polynomial. The last number is remainder which is written in the rectangle.

| | | | | |
|---|---|---|---|---|
| | 4 | – 20 | 17 | – 4 |
| 4 | ↓ | 16 | – 16 | 4 |
| | 4 | – 4 | 1 | $\boxed{0}$ |

From the third row the coefficients of the required quotient polynomial are 4 – 4 and 1. So the quotient is the polynomial $4x^2 - 4x + 1$. And the factorization of p(x) is, $p(x) = 4x^3 - 20x^2 + 17x - 4 = (x - 4) \times (4x^2 - 4x + 1)$.

***Self TestII*:**

## 14.5 : Roots of Polynomial equation:

**Definition 14.5.1 :** Polynomial equation: If $p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ is a polynomial, then the equation $p(x) = 0$ i.e. $a_0 + a_1x + a_2x^2 + \ldots + a_nx^n = 0$ is called a polynomial equation.

**Definition 14.5.2 :** Roots of Polynomial equation:

If $a_0 + a_1x + a_2x^2 + \ldots + a_nx^n = 0$ is a polynomial equation then the values of the variable x satisfying this equation are called roots of this equation.

For a polynomial equation $a_0 + a_1x + a_2x^2 + \ldots + a_nx^n = 0$, $x = \propto$ is a root, if

$a_0 + a_1 \propto + a_2 \propto^2 + \ldots + a_n \propto^n = 0$. If polynomial is of degree n, then that polynomial equation can have maximum n distinct roots.

Examples:

- For the polynomial equation $x^3 - 6x^2 + 11x - 6 = 0$ , its root is x = 2, because 2 satisfies this equation,

  i.e. $2^3 - 6 \times 2^2 + 11 \times 2 - 6 = 8 - 24 + 22 - 6 = 30 - 30 = 0$.

- For the polynomial equation $4x^3 - 20x^2 + 17x - 4 = 0$, its root is x = 4 because 4 satisfies this equation,

  i.e. $(4 \times 4^3) - (20 \times 4^2) + (17 \times 4) - 4 = 256 - 320 + 68 - 4 = 324 - 324 = 0$.

All roots of a polynomial equation p(x) = 0 can be found by different methods, one of the methods is of synthetic division. We studied this method earlier, but to find the roots by this method we need one factor of the polynomial p(x). If $(x - \alpha)$ is a factor of polynomial p(x) then $x = \alpha$ is a root of polynomial equation p(x )= 0, then the remaining roots can be found by synthetic division method.

To find a factor of the given polynomial tests of divisibility are useful. If these tests are not applicable then an integer root of p(x )= 0 can be from the divisors of the constant term in p(x ), this we can find by trial and error method.

## 14.5 : Tests of divisibility :

**1. Test for (x – 1 ):** If the sum of all the coefficients of a polynomial p(x) in x is zero, then (x – 1 ) is a factor of that polynomial. And x = 1 is a root of the polynomial equation p(x) = 0.

If x = 1 is a root of the polynomial equation then remaining roots can be found by synthetic division method.

Examples:

- Find a factor of $x^3 - 6x^2 + 9x - 4 = 0$

Let $p(x) = x^3 - 6x^2 + 9x - 4$. Sum of all coefficients in p(x) = 1 – 6 + 9 – 4 =0

$\therefore$ (x – 1 ) is a factor of p(x).

- Find a factor of $x^3 + x^2 - x - 1 = 0$.

Here the sum of all coefficients = 1 + 1– 1 – 1 =0.

$\therefore$ (x – 1 ) is a factor of $x^3 - x^2 + x - 1$.

**2. Test for (x +1 ):** If for a polynomial p(x), the sum of all the coefficients of even powers of x is equal to the sum of all the coefficients of odd powers of x then (x +1 ) is a factor of the polynomial p(x) . And x = –1 is a root of the polynomial equation p(x) = 0.

If x = –1 is a root of the polynomial equation then remaining roots can be found by synthetic division method.

Examples:

- Find a factor of $x^3 + 6x^2 + 9x + 4 = 0$

The sum of all the coefficients of even powers of x = 6 + 4 = 10

and the sum of all the coefficients of odd powers of x = 1 +9 =10

$\therefore$ (x + 1 ) is a factor of p(x).

- Find a factor of $x^3 + x^2 - x - 1 = 0$.

The sum of all the coefficients of even powers of x = 1 + (– 1)= 0

and the sum of all the coefficients of odd powers of x = 1 + (– 1)= 0

$\therefore$ (x + 1 ) is a factor of p(x).

- Find all roots of $x^3 - 6x^2 + 9x - 4 = 0$

<u>Solution:</u> Let $p(x) = x^3 - 6x^2 + 9x - 4$. Sum of all coefficients in p(x) = 1 – 6 + 9 – 4 =0 $\therefore$ By test for (x – 1 ) , it is a factor of p(x). Now to obtain the other factors use synthetic division. (x – 1 ) is a factor so x = 1 is a root of p(x) =0.

| | | | | |
|---|---|---|---|---|
| 1 | 1 | -6 | 9 | -4 |
| | ↓ | 1 | -5 | 4 |
| | 1 | – 5 | 4 | $\boxed{0}$ |

$\therefore$ the other factor of $p(x) = x^2 - 5x + 4$ .

So $p(x) = (x - 1) \times (x^2 - 5x + 4)$

Let $x^2 - 5x + 4 = q(x)$.

Sum of all coefficients in q(x) = 1 – 5 + 4 = 0. $\therefore$ By test for (x – 1 ) , it is a factor of q(x). Now to obtain the other factor use synthetic division again,

| | | | |
|---|---|---|---|
| 1 | 1 | -5 | 4 |
| | ↓ | 1 | -4 |
| | 1 | – 4 | $\boxed{0}$ |

$\therefore$ $q(x) = x^2 - 5x + 4 = (x - 1) \times (x - 4)$

Hence $p(x) = (x - 1) \times (x - 1) \times (x - 4)$

$\therefore$ The equation p(x) = 0 has the roots x = 1 , x = 1 and x = 4.

- Find all roots of $x^3 + x^2 - x - 1 = 0$

<u>Solution:</u> Let $p(x) = x^3 + x^2 - x - 1 = 0$.

Sum of all coefficients in p(x) = 1+ 1 – 1 – 1 =0 $\therefore$ By test for (x – 1 ) , it is a factor of p(x). Now to obtain the other factors use synthetic division. (x – 1 ) is a factor so x = 1 is a root of p(x) =0.

$$\begin{array}{c|cccc} 1 & 1 & 1 & -1 & -1 \\ & \downarrow & 1 & 2 & 1 \\ \hline & 1 & 2 & 1 & \boxed{0} \end{array}$$

$\therefore$ the other factor of $p(x) = x^2 + 2x + 1$ .

So $p(x) = (x - 1) \times (x^2 + 2x + 1)$

Let $q(x) = (x^2 + 2x + 1)$ .

Sum of all coefficients in q(x) = 1+ 2+ 1 = 4 $\neq$ 0. $\therefore$ The test for (x – 1 ) fails.
The sum of all the coefficients of even powers of x = 1 + 1 = 2
and the sum of all the coefficients of odd powers of x = 2

$\therefore$ (x + 1 ) is a factor of q(x) i.e. x = – 1 is a root of q(x) = 0. Now to obtain the other factor use synthetic division again,

$$\begin{array}{c|ccc} -1 & 1 & 2 & 1 \\ & \downarrow & -1 & -1 \\ \hline & 1 & 1 & \boxed{0} \end{array}$$

$\therefore$ $q(x) = x^2 + 2x + 1 = (x + 1) \times (x + 1)$

Hence $p(x) = (x - 1) \times (x + 1) \times (x + 1)$

$\therefore$ All the roots of $x^3 + x^2 - x - 1 = 0$, are x = 1 , x = – 1 and x = – 1.

## 14.6: Quadratic equations and their roots:

**Definition 14.6.1 :** Quadratic equation: If p(x ) = 0 is a polynomial equation , where p(x ) = is a polynomial of degree 2 , then such equation is called a quadratic equation.

In general an equation of the form $ax^2+bx+c = 0$, where a, b, c are real numbers and a ≠ 0, is a quadratic equation. It has two roots. These two roots of a quadratic equation can be found using above synthetic division method or by using following formulae.

If $ax^2+bx+c = 0$, where a, b, c are real numbers and a ≠ 0, is a quadratic equation, and α and β are its roots, then $\alpha = \dfrac{-b + \sqrt{b^2 - 4abc}}{2a}$ and $\beta = \dfrac{-b - \sqrt{b^2 - 4abc}}{2a}$.

Properties of roots of quadratic equation $ax^2+bx+c = 0$:

(i) If $\sqrt{b^2 - 4ac} = 0$, then the roots are real and equal.

(ii) If $\sqrt{b^2 - 4ac}$ is positive and a perfect square, then the roots are rational numbers and unequal.

(iii) If $\sqrt{b^2 - 4ac}$ is positive and not a perfect square, then the roots are irrational numbers and unequal.

(iv) If $\sqrt{b^2 - 4ac}$ is negative , then the roots are complex numbers and not real numbers.

Examples:

- Find the roots of the quadratic equation $x^2 - 6x + 9 = 0$,

Solution : In this equation the coefficients are a =1, b = – 6 and c= 9.

Letα and β be its roots,

then $\alpha = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$

i.e. $\alpha = \dfrac{-(-6) + \sqrt{6^2 - 4 \times 1 \times 9}}{2 \times 1} = \dfrac{6 + \sqrt{36 - 36}}{2} = \dfrac{6+0}{2} = \dfrac{6}{2} = 3$

and $\beta = \dfrac{-b - \sqrt{b^2 - 4abc}}{2a}$

i.e. $\beta = \dfrac{-(-6) - \sqrt{6^2 - 4 \times 1 \times 9}}{2 \times 1} = \dfrac{6 - \sqrt{36 - 36}}{2} = \dfrac{6-0}{2} = \dfrac{6}{2} = 3$.

$\therefore$ The roots of equation $x^2 - 6x + 9 = 0$, are $x = 3$ , $x = 3$ i.e. the roots are multiple roots.

- Find the roots of the quadratic equation $x^2 - 7x + 10 = 0$,

Solution : In this equation the coefficients are a =1, b = – 7 and c= 10.

Let $\alpha$ and $\beta$ be its roots,

then $\alpha = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ so,

$$\alpha = \frac{-(-7) + \sqrt{7^2 - 4\times1\times10}}{2\times1} = \frac{7+\sqrt{49-40}}{2} = \frac{7+\sqrt{9}}{2} = \frac{7+3}{2} = \frac{10}{2} = 5$$

and $\beta = \dfrac{-b - \sqrt{b^2 - 4abc}}{2a}$

$$\text{i.e. } \beta = \frac{-(-7) - \sqrt{7^2 - 4\times1\times10}}{2\times1} = \frac{7-\sqrt{49-40}}{2} = \frac{7-\sqrt{9}}{2} = \frac{7-3}{2} = \frac{4}{2} = 2.$$

$\therefore$ The roots of equation $x^2 - 7x + 10$, are $x = 5$ , $x = 2$.

- Find the roots of $3x^2 - x - 10 = 0$,

Solution : In this equation the coefficients are a =3, b = – 1 and c= – 10.

Let $\alpha$ and $\beta$ be its roots,

then $\alpha = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$

$$\text{i.e. } \alpha = \frac{-(-1) + \sqrt{1^2 - 4\times3\times(-10)}}{2\times3} = \frac{1+\sqrt{121}}{6} = \frac{1+11}{6} = \frac{12}{6} = 2$$

and $\beta = \dfrac{-b - \sqrt{b^2 - 4abc}}{2a}$

$$\text{i.e. } \beta = \frac{-(-1) - \sqrt{1^2 - 4\times3\times(-10)}}{2\times3} = \frac{1-\sqrt{121}}{6} = \frac{1-11}{6} = \frac{-10}{6} = \frac{-5}{6}.$$

$\therefore$ All the roots of $3x^2 - x - 10 = 0$, are $x = 2$ , $x = -\dfrac{5}{6}$.

## ***Self TestIII*:**

## 14.6: Summary for Unit 14:

In this unit learners studied the following topics in details:

**1.** The concept of polynomial in variable x over real numbers, its degree.

**2.** Different polynomials such as zero polynomial and constant polynomials.

**3.** Different operations on polynomials such as addition, difference, and multiplication, division of polynomials and synthetic division.

**4.** Roots of Polynomial equation and tests of divisibility.

**5.** Quadratic equations and the formula to find its roots.