Unit 4 : Exponents and Logarithms, Surds

4.0 Unit Objectives:

By the end of this Unit, learners should be able to:

- Understand Exponential form, integral exponents.
- State Laws of Exponents . and solve problems using Laws of Exponents.
- Simplify fractional exponents and surds.
- Explain Logarithms and Laws of Logarithms .
- Convert Logarithm to a different base.
- Perform complex calculations by applying Logarithms.

4.1 Unit Introduction:

Exponentiation is a mathematical operation used in many fields other than mathematics such as economics, biology, chemistry, physics and computer science. It is used in the calculations involving compound interest, population growth, chemical reaction kinetics, wave theory and public key cryptosystems.

Exponentiation where the exponent is a matrix is used for solving system of linear differential equation. Exponentiation is defined for even complex exponents. The special exponentiation function e^x is example of it. It is used to express the trigonometric functions as exponentiations.

Exponentiation is nothing but the multiplication by the same factor. The inverse of this operation gives us two operations which are extracting roots and taking logarithms.

The logarithm is perhaps the single, most useful arithmetic concept used in all sciences. Understanding logarithm is essential to understand many scientific ideas. Logarithms may be defined and introduced in several different ways. Previously when there were no calculators, arithmetic calculations involving large multiplications, divisions and powers were time consuming. Logarithms were invented to reduce the amount of work involved in such complex and difficult calculations. Even in these days of calculators and computers logarithm is still an important working tool in mathematics.

Another inverse operation of exponentiation is nothing but obtaining roots. In this unit we will study about special types of n-th roots of positive integers which are surds.

4.2 : Exponential form and Laws of Exponents

Exponentiation is a mathematical operation written as a^n, which corresponds to the repeated multiplication if n is a natural number. This operation is defined as, a^n =

$a\times a \times a \times a \times \ldots \times a$ (n times). In this case 'a' is called the base and 'n' is called the exponent.
Also, it is defined that $a^0 = 1$ and $a^1 = a$.

The exponent is generally shown as a superscript to the right of the base. The exponentiation a^n can be read as "a raised to the n^{th} power" or "a to the n^{th} power" or in short "a to the n". But a^2 is read as "a squared" and a^3 is read as "a cubed". As mentioned earlier any non zero number raised to the power 0 is 1 and any number raised to the power 1 is itself. The exponentiation, a^n is also defined when the exponent is a negative integer.

4.2.1 : Positive integer exponents:

The exponent is the number, that many times the base is multiplied together. Hence in general $a^n = a\times a \times a \times a \times \ldots \times a$, where n copies of 'a' are multiplied together.
If a is any real number then,
$a^0 = 1$, if a is not equal to 0.
$a^1 = a$.
$a^2 = a\times a$. It is also read as "a squared" because one can relate it with the area of a square whose sides are of length a.
$a^3 = a\times a \times a$. It is also read as "a cubed" because one can relate it with the volume of a cube whose sides are of length a.
Examples:

- $2^0 = 1$. $\therefore$ 2 to the power 0 is 1.
- $6^1 = 6$. $\therefore$ 6 to the power 1 is 6.
- $3^2 = 3 \times 3 = 9$. $\therefore$ 9 is 3 squared.
- $5^3 = 5 \times 5 \times 5 = 125$. $\therefore$ 125 is 5 cubed.
- $2^5 = 2\times 2 \times 2 \times 2 \times 2 = 32$. Here the base 3 is multiplied with itself 5 times as 5 is the exponent. $\therefore$ 32 is the 5^{th} power of 2 or 2 raised to the 5^{th} power.

4.2.2 : Negative integer exponents:

Any nonzero number raised to the –1 power is defined to be equal to the reciprocal.
$\therefore\ a^{-1} = 1 / a$ and hence
$a^{-n} = 1 / (a^n) = 1 / [\, a\times a \times a \times a \times \ldots \times a$ (n times multiplication)].
Examples:

- $2^1 = 2$. $\therefore\ 2^{-1} = 1 / 2$.
- $3^2 = 3 \times 3 = 9$. $\therefore\ 3^{-2} = 1 / 3^2 = 1 / (3 \times 3) = 1/ 9$.
- $5^3 = 5 \times 5 \times 5 = 125$. $\therefore\ 5^{-3} = 1/ 5^3 = 1 / (5 \times 5 \times 5) = 1/ 125$.

- $2^6 = 2\times 2 \times 2 \times 2 \times 2\times 2 = 64$.

  $\therefore 2^{-6} = 1/2^6 = 1/(2\times 2 \times 2 \times 2 \times 2\times 2) = 1/64$.

If the exponent n is positive then, then the n^{th} power of zero is zero i.e. $0^n = 0$. If the exponent is zero then 0^0 is undefined. Zero raised to the –1 power is also undefined because it is division by zero. So 0 raised to any negative power is undefined.

All the integer powers of 1 are 1 i.e. $1^n = 1$, for positive as well as negative n.

If the exponent is even number then, the power of –1 is 1 i.e. $(-1)^{2n} = 1$ and if the exponent is odd number then, the power of –1 is –1 i.e. $(-1)^{2n+1} = -1$.

In computer science, binary number system is very useful. In this number system positive powers of 2 are of much importance.

Exponentiation with base 10 is used to write very large or very small numbers. This method of writing numbers is called scientific notation method. To write speed of light, distances between stars, dimensions of bacteria or viruses numbers are written using scientific notations. E.g. the speed of light in vacuum is 299792458meters per second. Using scientific notations this number is written as 2.99792458×10^8 which is approximately equals to 2.998×10^8 or 3×10^8. Integer powers of 10 are also used to represent the numbers as $100000 = 10^5$, $100,000,000 = 10^8$ or $0.00001 = 10^{-5}$ etc.

4.2.3 : <u>Laws of Exponents:</u>

If a, b are real numbers and m and n are integers, then the following properties of exponents hold.

1. $a^0 = 1, a \neq 0$

Examples:

- $3^0 = 1$.
- $(1/2)^0 = 1$.

2. $a^m \times a^n = a^{m+n}$

Examples:

- $3^2 \times 3^4 = (3 \times 3) \times (3 \times 3 \times 3 \times 3) = 3^6 = 3^{2+4}$.
- $(\frac{1}{2})^4 \times (\frac{1}{2})^3 = (\frac{1}{2})\times(\frac{1}{2})\times(\frac{1}{2})\times(\frac{1}{2})\times(\frac{1}{2})\times(\frac{1}{2})\times(\frac{1}{2}) = (\frac{1}{2})^7$.

3. $(a^m)^n = a^{m\times n}$

Examples:

- $(3^2)^4 = 3^2 \times 3^2 \times 3^2 \times 3^2 = (3 \times 3)\times(3\times 3)\times(3\times 3)\times(3\times 3) = 3^8 = 3^{2\times 4}$.
- $[(\frac{1}{2})^4]^3 = (\frac{1}{2})^4 \times (\frac{1}{2})^4 \times (\frac{1}{2})^4 = (\frac{1}{2})^{12} = (\frac{1}{2})^{4\times 3}$.

4. $(a\times b)^m = a^m \times b^m$

Examples:

- $(2 \times 3)^4 = (2\times3)\times(2\times3)\times(2\times3)\times(2\times3) = (2\times2\times2\times2)\times(3\times3\times3\times3) = 2^4 \times 3^4$.
- $(7 \times 4)^3 = (7\times4)\times(7\times4)\times(7\times4) = (7 \times 7 \times 7)\times(4 \times 4 \times 4) = 7^3 \times 4^3$

5. $a^m / a^n = a^{m-n}$

Examples:

- $2^5 / 2^2 = (2 \times 2 \times 2 \times 2 \times 2) / (2 \times 2) = 2^3 = 2^{5-2}$.
- $4^9 / 4^5 = (4 \times 4 \times 4 \times 4 \times 4\times 4 \times 4 \times 4\times 4) / (4 \times 4\times 4 \times 4 \times 4) = 4^4 = 4^{9-5}$.

6. $(\frac{a}{b})^m = a^m / b^m$

Examples:

- $(2 / 3)^4 = (2/3)\times(2/3)\times(2/3)\times(2/3) = (2\times2\times2\times2) /(3\times3\times3\times3) = 2^4 / 3^4$.
- $(7 \times 5)^3 = (7\times5)\times(7\times5)\times(7\times5) = (7 \times 7 \times 7)\times(5 \times 5 \times 5) = 7^3 \times 5^3$

7. $a^{-m} = 1 / a^m \quad , a \neq 0$

Examples:

- $2^{-3} = 1 / 2^3 = (\frac{1}{2})\times(\frac{1}{2})\times(\frac{1}{2}) = (\frac{1}{2})^3$.
- $4^{-5} = 1 / 4^5 = (\frac{1}{4})\times(\frac{1}{4})\times(\frac{1}{4})\times(\frac{1}{4})\times(\frac{1}{4}) = (\frac{1}{4})^5$.

Here we must note that the multiplication of real numbers is a commutative and associative operation, but exponentiation is not both commutative as well as associative operation.

Multiplication of real numbers is a commutative operation i.e. $a \times b = b \times a$.

But $a^b \neq b^a$. e.g. $3^4 = 3 \times3 \times3 \times3 = 81$ and $4^3 = 4 \times 4 \times 4 = 64$.

$\therefore$ exponentiation is not commutative operation.

Multiplication of real numbers is an associative commutative operation

i.e. $a \times (b \times c) = (a \times b) \times c$.

But $a^{(b^c)} \neq (a^b)^c$, $\therefore$ exponentiation is not associative operation.

e.g. 3 raised to (2^4) th power is 3 raised to 16th power = 3^{16} = 43046721

and 3^2 raised to 4th power is 9 raised to 4th power $=9^4 =6561$.

$\therefore 3^{(2^4)} \neq (3^2)^4$.

❖ ***Self Test I*: Select the correct alternative from the given alternatives.**

1. What is the exponential form of $4\times 4 \times 4 \times 4 \times 4\times 4$?

(a) 4^3 **(b)** 4^4 **(c)** 4^6 **(d)** 4^8.

2. What is the exponential form of $1 / (5\times 5 \times 5 \times 5)$?

(a) 5^4 **(b)** 5^{-4} **(c)** 4^5 **(d)** 4^{-5}.

3. What is the exponential form of $-5 \times -5 \times -5 \times -5$?

(a) $-(5^4)$ **(b)** 5^{-4} **(c)** $(-5)^4$ **(d)** 4^{-5}.

4. What is the exponential form of $(1/3) \times (1/3) \times (1/3) \times (1/3) \times (1/3)$?

(a) $-(3)^5$ **(b)** 3^{-4} **(c)** $(1/3)^5$ **(d)** 5^{-3}.

5. What is the value of a^0 if $a \neq 0$?

(a) 0 **(b)** 1 **(c)** undefined quantity **(d)** a .

6. What is the value of 0^0 ?

(a) 0 **(b)** 1 **(c)** undefined quantity **(d)** -1 .

7. What is the value of $(-1)^5$?

(a) 0 **(b)** 1 **(c)** undefined quantity **(d)** -1 .

8. What is the value of 10^{-4}?

(a) 10000 **(b)** 0.0001 **(c)** 0.00001 **(d)** 10 .

9. What is the simplification of $(3^7 \times 3^{-2} \times 3^0) / 3^4$?

(a) 1 **(b)** 3 **(c)** 3^2 **(d)** 0.

10. What is the simplification of $(2^3 \times 2^{-6} \times 2^6 \times 2^{-7}) / (2^4 \times 2^{-5})$?

(a) 2^0 **(b)** 2^4 **(c)** 2^3 **(d)** 2^{-3}.

4.3: Fractional exponents and surds:

One inverse operation of exponentiation is nothing but obtaining roots of a number. If a and b are two numbers such that, b is exponential form a^n , i. e. $b = a^n$, then a is defined to be an n-th root of b. Equivalently it is also written as $a = b^{1/n}$. So the nth roots of a number involve fractional exponents. These are also denoted using radical symbol or the root symbol i.e. if $a = b^{1/n}$ then we write $a = \sqrt[n]{b}$. We will study the n-th roots of positive real numbers only.

Generally a second root of a number is called as its square root and a third root is called as a cube root of a number.

Examples:

- $2^5 = 32 \therefore 2$ is a 5^{th} root of 32. It is written as $2 = (32)^{1/5} = \sqrt[5]{32}$.

- $3^2 = 3 \times 3 = 9$. $\therefore$ 3 is a square root of 9 , hence $3 = 9^{1/2}$.

  Also $(-3)^2 = -3 \times -3 = 9$. $\therefore$ -3 is another square root of 9.

  Symbolically is written as $\sqrt{9} = \pm 3$.
- $5^3 = 5 \times 5 \times 5 = 125$. $\therefore$5 is a cube root of 125 or $5 = (125)^{1/3}$.

  Symbolically is written as $5 = \sqrt[3]{125}$.

The earlier studied laws of exponents are applicable even to fractional exponents. a special class numbers involving fractional exponents is of surds. Surds are numbers written in radical form which are irrational numbers of a particular type.

4.3.1: Laws of fractional exponents:

If a, b are positive real numbers and m and n rational numbers, then we have,

1. $a^m \times a^n = a^{m+n}$
2. $(\frac{a}{b})^m = a^m / b^m$
3. $(a^m)^n = a^{m \times n}$

Note:

1. A number which can be written in the form $\frac{p}{q}$ where p and q are integers and $q \neq 0$, is called a rational number.
2. The numbers which are not rational numbers are called as irrational number. For irrational numbers the decimal representation is non terminating and non recurring.

Examples: All integers are rational numbers. Also the numbers $\frac{1}{2}, \frac{4}{7}, \frac{25}{125}$ and $\sqrt{4}$ etc. are rational numbers. But $\sqrt{2}, \sqrt{3}, \sqrt[4]{5}$ etc. are not rational numbers. Also the constant π is an irrational number, where π is the ratio of circumference of any circle to its diameter.

Definition: 4.3.1: Surd: A surd is a number $\sqrt[n]{a} = a^{1/n}$ if and only if

1. it is an irrational number ,
2. $n \neq 1$ is a natural number and a is positive rational number .

Examples:

- $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}$ are all surds but $\sqrt{4}$ is not a surd as it equals to 2, which is a rational number.
- $\sqrt[3]{2}$, $\sqrt[5]{8}$, $\sqrt[4]{20}$ are all surds but $\sqrt[3]{125}$ is not a surd as it equals to 5 which is a rational number.

- $\sqrt{5+\sqrt{3}}$ is not a surd as 5+$\sqrt{3}$ is not a rational number, also $\sqrt{\pi}$ is not a surd because π is an irrational number.

Using rules of operations of radicals, we can determine that whether a given number is a surd or not.

Examples:

- $\sqrt{10}\times\sqrt{90}=\sqrt{900}=\sqrt{30\times 30}=\left(\sqrt{30}\right)^2$ = 30 it is a rational number. Rational numbers are not suds. $\therefore$ $\sqrt{10}\times\sqrt{90}$ is not a surd . But $\sqrt{10}$ and $\sqrt{90}$ are surds.
- $\sqrt[3]{128}=\sqrt[3]{4\times 4\times 4\times 2}=\sqrt[3]{64}\times\sqrt[3]{2}=4\sqrt[3]{2}$ is a surd as it is an irrational number which satisfy the definition of a surd .
- $\sqrt{\sqrt{7}}$ = $(7^{1/2})^{1/2}$ = $7^{1/4}$ = $\sqrt[4]{7}$ it is a surd, by the definition of a surd.

4.3.2: Rules of operations with surds:

1. $(\sqrt[n]{a})^n$ = a.

Examples:

- $(\sqrt[3]{2})^3$ = 2.
- $(\sqrt[5]{4})^5$ = 4.

2. $\sqrt[n]{ab}=\sqrt[n]{a}\times\sqrt[n]{b}$, a ≥ 0 and b ≥ 0.

Example:

- $\sqrt[3]{20}=\sqrt[3]{4}\times\sqrt[3]{5}$

3. $\sqrt[n]{\dfrac{a}{b}}=\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$, a ≥ 0 and b> 0.

Example:

- $\sqrt[3]{\dfrac{4}{5}}=\dfrac{\sqrt[3]{4}}{\sqrt[3]{5}}$.

4. $\sqrt[n]{a}^{\,m}=(\sqrt[n]{a})^m=(a^{1/n})^m=a^{m/n}$

Examples:

- $\sqrt[3]{2}^{\,6}=(\sqrt[3]{2})^6=(2^{1/3})^6=2^{6/3}=2^2=4$.
- $\sqrt[3]{5}^{\,4}=(\sqrt[3]{5})^4=(5^{1/3})^4=5^{4/3}$.

5. $\sqrt[m]{\sqrt[n]{a}}=\sqrt[mn]{a}=\sqrt[n]{\sqrt[m]{a}}$

Examples:

- $\sqrt[2]{\sqrt[3]{7}}=\sqrt[6]{7}=\sqrt[3]{\sqrt[2]{7}}$

- $\sqrt[3]{\sqrt[5]{21}} = \sqrt[15]{21} = \sqrt[5]{\sqrt[3]{21}}$

Definition: 4.3.2: Order of a surd: In the surd $b\sqrt[n]{a}$, b is called coefficient of the surd, the index n is called the order of the surd and a is called radicand . When the coefficient of a surd is not written, it is assumed to be 1. When the order n of a surd is not mentioned it is taken as 2.

Examples:

- In the surd $5\sqrt[3]{2}$, the coefficient of the surd is 5, the order of the surd is 3 and the radicand is 2.
- In the surd $\sqrt[4]{5}$ the order of the surd is 4 and the radicand is 5 and coefficient is 1.

4.3.2 : Forms of surds:

Definition: 4.3.2: Pure Surd : A surd of the form $\sqrt[n]{a}$ is called a pure surd.

Examples: Following are pure surds:

- $\sqrt{10}$.
- $\sqrt[3]{12}$.
- $\sqrt[4]{7}$.

Definition: 4.3.3: Mixed Surd : A surd of the form $b\sqrt[n]{a}$ is called a mixed surd where b is a rational number and b $\neq$1.

Examples: Following are mixed surds:

- $5\sqrt{10}$.
- 2 $\sqrt[3]{12}$.
- 8 $\sqrt[4]{7}$.

Note: A mixed surd may be expressed as a pure surd as well as a pure surd may be expressed as a mixed surd.

Examples:

- 5 $\sqrt[3]{10} = \sqrt[3]{5\times5\times5\times10} = \sqrt[3]{1250}$
- $\sqrt{112} = \sqrt{16\times7} = \sqrt{4\times4}\times\sqrt{7} = 4\sqrt{7}$.

Definition: 4.3.4: Similar or like surds : The surds of the form $p\sqrt[n]{a}$ and $q\sqrt[n]{a}$ are called similar surds or like surds, where p and q are rational numbers.

Examples: Following are similar surds

- 5 $\sqrt[3]{10}$, $\sqrt[3]{10}, 6\sqrt[3]{10}$
- $\sqrt{7}$, $100\sqrt{7}, 5\sqrt{7}, 4\sqrt{7}$

Definition: 4.3.4: Simplest form of a Surd : A surd $\sqrt[n]{a}$, is said to be in its simplest form if

1. the radicand "a " has no divisor which is n^{th} power of a rational number.
2. the radicand "a " is not a fraction and
3. "n" is the least such power.

Examples:

- $\sqrt[3]{1250} = \sqrt[3]{125\times 10} = \sqrt[3]{125}\times\sqrt[3]{10} = 5\sqrt[3]{10}$, $\therefore$ The surd $\sqrt[3]{1250}$ has the simplest form $5\sqrt[3]{10}$
- $\sqrt[4]{1875} = \sqrt[4]{625\times 3} = \sqrt[4]{25\times 25\times 3} = \sqrt[4]{5\times 5\times 5\times 5}\times\sqrt[4]{3} = 5\sqrt[4]{3}$

$\therefore$ The surd $\sqrt[4]{1875}$ has the simplest form $5\sqrt[4]{3}$.

❖ ***Self Test II*: Select the correct alternative from the given alternatives.**

1. What is the index of the surd $\sqrt[3]{15}$?

(a) 15 **(b)** 2 **(c)** 3 **(d)** 5.

2. What is the radicand of the surd $\sqrt[3]{15}$?

(a) 15 **(b)** 2 **(c)** 3 **(d)** 5.

3. Which of the following radicals is a surd?

(a) $\sqrt[3]{125}$ **(b)** $\sqrt[4]{15}$ **(c)** $\sqrt{25}$ **(d)** $\sqrt[4]{81}$.

4. Which of the following is a surd?

(a) $\sqrt[3]{1+\sqrt{8}}$ **(b)** $\sqrt[3]{8}$ **(c)** $\sqrt{\sqrt{5}}$ **(d)** $\sqrt[4]{16}$.

5. Which of the following is a pure surd?

(a) $\sqrt[3]{125}$ **(b)** $\sqrt[4]{15}$ **(c)** $2\sqrt{5}$ **(d)** $5\sqrt[4]{81}$.

6. Which of the following is a mixed surd?

(a) $\sqrt[3]{125}$ **(b)** $\sqrt[4]{15}$ **(c)** $2\sqrt{5}$ **(d)** $5\sqrt[4]{81}$.

7. Which of the following is a pair of similar surds?

(a) $\sqrt[3]{12}, \sqrt{12}$ **(b)** $3\sqrt[5]{7}, 9\sqrt[5]{7}$ **(c)** $2\sqrt{5}, \sqrt[4]{5}$ **(d)** $5\sqrt[4]{8}, 4\sqrt{8}$.

8. What is the simplest form of the surd $\sqrt[3]{135}$?

(a) $\sqrt[3]{5\times 27}$ **(b)** $3\sqrt[3]{27}$ **(c)** $\sqrt[3]{5\times 3\times 3\times 3}$ **(d)** $\sqrt[3]{135}$.

9. What is the simplest form of the surd $\sqrt{\dfrac{343}{45}}$?

(a) $\sqrt{\frac{49\times7}{9\times5}}$ **(b)** $\sqrt{\frac{49\times7}{9\times5}}$ **(c)** $\frac{7}{3}\sqrt{\frac{7}{5}}$ **(d)** $\frac{7}{15}\sqrt{35}$.

10. What is the simplification of $3\sqrt{8}+\sqrt{32}-5\sqrt{2}$?

(a) $3\sqrt{4\times2}+\sqrt{16\times2}-5\sqrt{2}$ **(b)** $-2\sqrt{20\times2}$ **(c)** $5\sqrt{2}$ **(d)** $-5\sqrt{2}$.

4.4 : Logarithms and Laws of Logarithms

4.4.1 :Logarithms:

Definition 4.4.1:Logarithm: If a and b are two positive real numbers such that, a is exponential form b^n , i. e. $a = b^n$ and $b \neq 1$, then $n = \log_b a$. i.e. the logarithm of b to the base a is n.

So the logarithmic equivalent form of the exponential equation $a = b^n$ is $n = \log_b a$.

Examples:

- $7^0 = 1 \therefore 0 = \log_7 1$.
- $4^1 = 4 \therefore 1 = \log_4 4$.
- $2^5 = 32. \therefore 5 = \log_2 32$ i.e. the logarithm of 32 to the base 2 is 5.
- $5^3 = 5 \times 5 \times 5 = 125. \therefore 3 = \log_5 125$, i.e. the logarithm of 125 to the base 5 is 3.
- $10^4 = 10000. \therefore 4 = \log_{10} 10000$, i.e. the logarithm of 10000 to the base 10 is 4.

Note that:

1. $\log_a 1 = 0$ for all $a > 0$, because $a^0 = 1$.

2. $\log_a a = 1$ for all $a > 0$, because $a^1 = a$.

3. Logarithms $\log_b a$ are defined only for positive values of a and b, when $b \neq 1$.

4. The logarithm of a to the base 10 i.e. $\mathbf{\log_{10} a}$ is also called as common logarithm and it is written as log a.

Examples:

- Since $10 = 10^1. \therefore \log 10 = 1$.
- Since $1000 = 10^3. \therefore \log 10000 = 3$.
- Since $0.1 = 10^{-1}. \therefore \log 0.1 = -1$.
- Since $0.0001 = 10^{-4}. \therefore \log 0.0001 = -4$.

5. The logarithm of a to the base e is also called as natural log of a, where e has approximate value 2.7182 and it is an irrational number.

Examples:

- Since e = e^1. $\therefore$ $\log_e e = 1$, similarly $\log_e e^2 = 2$ etc.
- Since $1 / (e^2) = e^{-2}$. $\therefore$ $\log_e (1 / (e^2)) = -2$.

4.4.2 : Laws of Logarithms:

If a, b are positive real numbers , x is a positive real number such that $x \neq 1$,and n is a real number, then the following properties of Logarithms hold.

1. $\log_x (a \times b) = \log_x a + \log_x b$.

Example: $\log_5 (25 \times 125) = \log_5 25 + \log_5 125 = \log_5 5^2 + \log_5 5^3 = 2 + 3 = 5$.

2. $\log_x (\frac{a}{b}) = \log_x a - \log_x b$.

Example: $\log_5 (\frac{125}{25}) = \log_5 125 - \log_5 25 = \log_5 5^3 - \log_5 5^2 = 3 - 2 = 1$.

3. $\log_x (a^n) = n \times \log_x a$.

Example: $\log_2 64 = \log_2 (4^3) = 3 \times \log_2 4 = 3 \times 2 = 6$.

4. $\log_x x^a = a$.

Example: $\log_5 5^2 = 2$ and $\log_5 5^3 = 3$.

5. $x^{(\log_x a)} = a$.

Example: $x^{(\log_x a)}$ $\log_5 5^2 = 2$ and $\log_5 5^3 = 3$.

4.4.3 : Antilogarithms:

The process of finding antilogarithm is just the reverse of finding a logarithm. We know that $4 = \log_{10} 10000$, $\therefore$ $\text{antilog}_{10} 4 = 10000$. Antilogarithm is found using the same base as that of the corresponding logarithm. While doing complicated calculations we need antilogarithms to find final anser.

4.5 : Conversion to a different Base:

While doing calculations it is frequently needed to change the logarithm of a number to a different base. The following rule is used for conversion of the logarithm of a number to a different Base :

$$\log_b a = \frac{\log_x a}{\log_x b}.$$

It is more common to use common logarithm (i.e. to the base 10) in place of natural logarithm(i.e. to the base e). In changing the base from e to 10 and vice versa we need the values of $\log_{10} e$ and $\log_e 10$, which are

$\log_{10} e \approx 0.4343$ and $\log_e 10 \approx 2.3026$.

Examples:

- $\log_e 100 = \dfrac{\log_{10} 100}{\log_{10} e} = \dfrac{2}{0.4343} = 4.6051$
- We have seen above that $\log_2 64 = 6$. Also

$$\log_2 10 = \frac{\log_e 10}{\log_e 2} = \frac{2.3026}{0.6931} \approx 3.3222$$

$$\therefore \log_{10} 64 = \frac{\log_2 64}{\log_2 10} = \frac{6}{3.3222} \approx 1.8060$$

4.6 : Application of Logarithms in complex calculations:

Logarithms and antilogarithms can be used to do laborious and tedious calculations involving very large or very small numbers. For such calculations laws of logarithms, logarithm table and antilogarithm table is used. Generally common logarithm or logarithm to the base 10 is used in calculations. It is common practice to write log x in place of $\log_{10} x$.

Examples:

1. Find $(1.57)^5$.

Solution: let $x = (1.57)^5$.

$\therefore$ $\log(x) = \log((1.57)^5) = 5 \times \log(1.57) = 5 \times 0.1959 = 0.9795$

$\therefore$ $\times$ = antilog (0.9795) = 9.539.

2. Find $\sqrt[3]{32}$.

Solution: let $x = \sqrt[3]{32} = (32)^{1/3}$.

$\therefore$ $\log(x) = \log((32)^{1/3}) = \frac{1}{3} \times \log(32) = \frac{1}{3} \times 1.5051 = 0.5017$

$\therefore$ x = antilog (0.5017) = 3.1746.

3. Calculate $(45.4)^2 / ((3.2)^2 \times (5.6)^3)$.

Solution: let $x = (45.4)^2 / ((3.2)^2 \times (5.6)^3)$.

$\therefore$ $\log(x) = \log(45.4)^2 / ((3.2)^2 \times (5.6)^3)$.

$= \log(45.4)^2 - \log((3.2)^2 \times (5.6)^3)$

$= \log(45.4)^2 - \{\log(3.2)^2 + \times \log(5.6)^3\}$

$= 2 \times \log(45.4) - \{2 \times \log(3.2) + 3 \times \log(5.6)\}$

$= 2 \times 1.6571 - \{2 \times 0.5051 + 3 \times 0.7482\}$

$= 0.0594$

$\therefore$ x $= \text{antilog}\ (0.0594) = 1.1465.$

<u>*Self Test III*</u>: Select the correct alternative from the given alternatives.

1. What is the logarithmic form of the exponential equation $7^3 = 343$?

(a) $3 = \log_7 343$ **(b)** $7 = \log_3 343$ **(c)** $3 = \log_{10} 343$ **(d)** $3 = \log_e 343$.

2. What is the logarithmic form of the exponential equation $\sqrt{16} = 4$?

(a) $\frac{1}{2} = \log 16$ **(b)** $16 = \log_{1/2} 4$ **(c)** $4 = \log_2 16$ **(d)** $\frac{1}{2} = \log_{16} 4$.

3. What is the exponential form of the logarithmic equation $\log_{10} 1000 = 3$?

(a) $3^{10} = 1000$ **(b)** $\sqrt[3]{1000} = 3$ **(c)** $10^3 = 1000$ **(d)** $10^{-3} = 1000$.

4. What is the exponential form of the logarithmic equation $-2 = \log_3 (\frac{1}{9})$?

(a) $3^9 = -2$ **(b)** $\sqrt[2]{\frac{1}{9}} = 3$ **(c)** $\frac{1}{9} = 2^{-3}$ **(d)** $\frac{1}{9} = 3^{-2}$.

5. Which of the following is the common logarithm of 1000000?

(a) 2 **(b)** 4 **(c)** 6 **(d)** 8.

6. Which of the following is the common logarithm of 0.000001?

(a) 6 **(b)** – 5 **(c)** – 6 **(d)** – 4.

7. What is the value of x if $\log_e x = 3$?

(a) $e = 2.7182$ **(b)** $e^2 = 7.3886$ **(c)** $e^3 = 20.0837$ **(d)** $e^{-2} = 0.1353$.

8. What is the value of $\log_e 12$, if $\log_{10} 12 = 1.0792$?

(a) 0.4686 **(b)** 2.4849 **(c)** 0.4343 **(d)** 2.3026.

9. What is the value of $\log_2 6$, if $\log_{10} 6 = 0.7782$ and $\log_{10} 2 = 0.3010$?

(a) 2.5853 **(b)** 2.4849 **(c)** 0.4343 **(d)** 2.3026.

10. What is the value of $\sqrt[3]{\frac{3 \times 71.34}{7.284}}$?

(a) 29.3822 **(b)** – 3.0857 **(c)** 3.0857 **(d)** 0.7071.

4.7 Summary for Unit 4

In this unit learners studied the following topics in details:

1. What are Exponential forms, and Laws of Exponents for real numbers.
2. Positive integer exponents and Negative integer exponents and Laws of Exponents.
3. Fractional exponents , surds and Rules of operations with surds
4. Order of a surd, Pure surd, Mixed surd, and Similar or like surds and simplest form of a Surd.
5. Logarithms and Laws of Logarithms, Antilogarithms.
6. How to convert logarithm of a number to a different Base.
7. How to use Logarithms in doing complex calculations.