UNIT 11 : Matrices and Determinants

11.0 : Unit Objectives:

By the end of this Unit, learners should be able to:

- Understand the concept of matrix.
- Describe and discuss different types of matrices.
- Perform different operations on matrices such as addition, subtraction, scalar multiplication, multiplication etc.
- Find inverse of a matrix.
- Understand concept of determinant.
- Describe and discuss different properties of determinant.
- Evaluate determinant of a matrix.

11.1 : Unit Introduction:

The theory of matrices (single matrix) is of great importance in many branches of higher mathematics such as astronomy, mechanics, nuclear physics and aerodynamics etc. Matrices are used to describe linear equations and to record data that depend on multiple parameters. Hence matrices are widely used in computer science. They can be added, multiplied, and decomposed in various ways, which also makes them a key concept in the field of linear algebra.

In elementary algebra we mainly deal with single numbers. These numbers are combined by various operations to obtain other numbers. But in some branches of algebra we need to consider a set of numbers. For example in plane geometry the Cartesian coordinates of a point are given by a pair of two numbers these numbers can be represented as a single entity using matrix. A matrix is only arrangements of numbers but it is treated as a single entity rather than a collection of numbers. In this unit we will get introduced with elementary concepts about matrices.

11.2 : Matrices:

In mathematics, a matrix (plural matrices) is an arrangement of numbers into rows and columns. The horizontal lines in a matrix are called rows and the vertical lines are called columns.

Definition 11.2.1: Matrix: A matrix is a set of numbers which are arranged into rows and columns.

A matrix with m rows and n columns is called an $m \times n$ matrix. It is commonly said that an $m \times n$ matrix is of order $m \times n$. The order of a matrix is always given with the number of rows first, then the number of columns.

The entry that lies in the i-th row and the j-th column of a matrix is typically referred to as the (i, j)th entry of the matrix. Again, the row is always noted first and then the column. Generally capital letters are used to denote matrices.

We often write $A = \left[a_{ij}\right]_{m \times n}$ to define an $m \times n$ matrix A. In this case, the entries $a_{i,j}$ are defined separately for all integers $1 \le i \le m$ and $1 \le j \le n$.

We can denote a general 3×4 matrix as, $A = \left[a_{ij}\right]_{3 \times 4}$ then this matrix can also be represented as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}_{3 \times 4}$$

Examples:

- $A = [1 \quad 2 \quad 3]$ is a matrix with 1 row and 3 columns. $\therefore$ A has an order 1×3.
- $B = \begin{bmatrix} 1 & 5 & 2 \\ 5 & 2 & 9 \\ 2 & 9 & 8 \end{bmatrix}$ is a matrix with 3 row and 3 columns. $\therefore$ A has an order 3×3.
- $P = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is a 2×4 matrix.
- $I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is a 4×4 matrix.

11. 3: Types of Matrices:

1. Row matrix: A matrix having only one row is called a row matrix. If P is a row matrix then its order is of the form $1 \times n$, where n is the number of columns in this matrix.

Example:

- $P = [1 \quad 5 \quad 7 \quad -4]$ is a row matrix as it has 1 row and 4 columns.

  $\therefore$ P has order 1×4.

2. Column matrix: A matrix having only one column is called a column matrix. If A is a column matrix then its order is of the form m × 1, where m is the number of columns in this matrix.

Example:

- $A = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is a column matrix as it has 3 rows and 1 column.

  $\therefore$ A has an order 3 × 1.

3. Square matrix: If the number of rows in a matrix is equal to the number of columns in it, then that matrix is called a square matrix.

If matrix $A_{m \times n}$ is a square matrix then m = n. So its order can also be written as n × n.

For a general square matrix $A = \left[a_{ij}\right]_{n \times n}$, the entries a_{11}, a_{22}, a_{33}, …, a_{nn} are called the diagonal elements.

Examples:

- $A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$ is a matrix with 2 rows and 2 columns. $\therefore$ A is a square matrix.

In this matrix the diagonal elements are $a_{11} = 2$ and $a_{22} = 2$.

- $B = \begin{bmatrix} 1 & 4 & 2 \\ 5 & 9 & 9 \\ 2 & 11 & -4 \end{bmatrix}$ is a 3 × 3 matrix, so it is a square matrix. In this matrix

the diagonal elements are $b_{11} = 1$, $a_{22} = 2$ and $a_{33} = 8$.

- $I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is a square matrix, in which all the diagonal elements are

equal to 1.

4. Zero or null matrix: If every element in a matrix is equal to zero, then that matrix is called a zero matrix or a null matrix. It is denoted by 0.

Examples:

- $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ ∴ A is a square matrix. In this matrix all elements are 0, and so it is a null matrix of order 2 × 2. It is also denoted as $0_{2\times2}$.
- N = $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is a null matrix, of order 3 × 5.

5. <u>Unit or Identity matrix:</u> A square matrix in which all diagonal elements are equal to 1 and all non- diagonal elements are equal to 0, is called an identity matrix. An identity matrix of order n × n is denoted by I_n.

Examples:

- $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is an identity matrix of order 2 × 2.
- $I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is an identity matrix of order 4 × 4.

6. <u>Diagonal matrix:</u> A square matrix, in which all non-diagonal elements are equal to 0, is called a diagonal matrix.

Note that, in a diagonal matrix all non-diagonal elements are equal to 0 and some or all of the diagonal elements may be equal to 0.

Examples:

- An identity matrix I_n is always a diagonal matrix .
- A square matrix which is a null matrix is a diagonal matrix.
- D = $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ is a diagonal matrix of order4 × 4.
- $A = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$ is a diagonal matrix of order 2 × 2.

7. <u>Triangular matrix:</u>

(i) <u>Upper triangular matrix</u> : A square matrix, in which all the elements below the main diagonal are equal to 0, is called an upper triangular matrix.

(ii) Lower triangular matrix: A square matrix, in which all the elements above the main diagonal are equal to 0, is called a lower triangular matrix.
Note that:
1. A triangular matrix is either a lower triangular or an upper triangular matrix.
2. A diagonal matrix and an identity matrix are both lower triangular and upper triangular matrix.

Examples:

- P = $\begin{bmatrix} 1 & 5 & 0 & -2 \\ 0 & 2 & 4 & 1 \\ 0 & 0 & 3 & 9 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ is an upper triangular matrix.
- $Q = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 5 & 30 \end{bmatrix}$ is a lower triangular matrix.
- $A = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$ is both lower triangular and upper triangular matrix.

8. Transpose of a matrix: If A is a matrix of order m × n, then the transpose of matrix A is denoted by A^t or A', is the matrix obtained by interchanging the rows and the columns of matrix A. Order of A^t is n × m.

For a general square matrix $A = \left[a_{ij}\right]_{m \times n}$,its transpose is $A^t = \left[a_{ji}\right]_{n \times m}$.

Note that,
1. Transpose of a row matrix is a column matrix.
2. Transpose of a column matrix is a row matrix.
3. Transpose of an identity matrix is itself.

Examples:

- $A = [1 \quad 2 \quad 3]$ is a row matrix with order 1 × 3.

  ∴ A^t is a column matrix with order 3 × 1 and $A^t = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

- $P = \begin{bmatrix} -1 & 3 & 2 & 1 \\ 6 & 4 & 0 & 5 \end{bmatrix}$ is a matrix with order 2 × 4.

$\therefore$ P^t is a matrix with order 4 × 2 and $P^t = \begin{bmatrix} -1 & 6 \\ 3 & 4 \\ 2 & 0 \\ 1 & 5 \end{bmatrix}$

- $B = \begin{bmatrix} 1 & 5 & 2 \\ 5 & 2 & 9 \\ 2 & 9 & 8 \end{bmatrix}$ is a matrix with order 3 × 3.

$\therefore$ B^t is a matrix with order 3 × 3 and $B^t = \begin{bmatrix} 1 & 5 & 2 \\ 5 & 2 & 9 \\ 2 & 9 & 8 \end{bmatrix}$

9. <u>Symmetric matrix</u>: A square matrix $A = \left[a_{ij}\right]_{n \times n}$ is a symmetric matrix, if $a_{ij} = a_{ji}$ for all i and j.

Note that,

Note that,

1. A square matrix A is symmetric matrix, if and only if $A = A^t$.
2. Every identity matrix is a symmetric matrix.
3. Every null matrix of order n × n is a square matrix for natural number n>1.
4. Every diagonal matrix is a symmetric matrix.

Examples:

- If $A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ then $A^t = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$, $\therefore$ A is a symmetric matrix.

- If $B = \begin{bmatrix} 1 & 5 & 2 \\ 5 & 2 & 9 \\ 2 & 9 & 8 \end{bmatrix}$ then $B^t = \begin{bmatrix} 1 & 5 & 2 \\ 5 & 2 & 9 \\ 2 & 9 & 8 \end{bmatrix}$. $\therefore$ B is a symmetric matrix.

- If $P = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 7 & 3 \\ 1 & 3 & 8 \end{bmatrix}$ then $P^t = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 7 & 3 \\ 1 & 3 & 8 \end{bmatrix}$. $\therefore$ P is a symmetric matrix.

Self Test I:

11.4 : Algebra of matrices:

11.4.1 : Equality of matrices:
Two matrices are equal if they are of the same order and if the corresponding elements are equal. So, if $A = \left[a_{ij}\right]_{m \times n}$ and $B = \left[b_{ij}\right]_{m \times n}$ are two matrices then, A = B if and only if $a_{ij} = b_{ij}$ for all i and j. Note that, matrices of different orders can not be equal.

11.4.2 : Addition of matrices:
If two matrices A and B are of the same order then they can be added and their addition denoted by A + B is the matrix of the same order which is obtained by adding the corresponding elements of A and B.

Definition 11.4.1: <u>Addition of matrices</u>: If $A = \left[a_{ij}\right]_{m \times n}$ and $B = \left[b_{ij}\right]_{m \times n}$ are two matrices then, $A + B = \left[a_{ij} + b_{ij}\right]_{m \times n}$.

Note that, matrices of different orders can not be added.

Examples:

- If $A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then $A + B = \begin{bmatrix} 2+1 & 2+2 \\ 2+3 & 2+4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$

- If $C = \begin{bmatrix} 1 & 5 & 2 \\ 5 & 2 & 9 \\ 2 & 9 & 8 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 7 & 3 \\ 1 & 3 & 8 \end{bmatrix}$, then $C + D = \begin{bmatrix} 3 & 5 & 3 \\ 5 & 9 & 12 \\ 3 & 12 & 16 \end{bmatrix}$.

- If $P = \begin{bmatrix} -1 & 3 & 2 & 1 \\ 6 & 4 & 0 & 5 \end{bmatrix}$ and $Q = \begin{bmatrix} 3 & 5 & -2 & 1 \\ 7 & 9 & 4 & 5 \end{bmatrix}$, then

$$P + Q = \begin{bmatrix} -1+3 & 3+5 & 2+(-2) & 1+1 \\ 6+7 & 4+9 & 0+4 & 5+5 \end{bmatrix} = \begin{bmatrix} 2 & 8 & 0 & 2 \\ 13 & 13 & 4 & 10 \end{bmatrix}.$$

<u>**Properties of addition of matrices:**</u>

1. Addition of matrices is commutative i.e. A + B = B + A.
2. Addition of matrices is associative i.e. A + (B + C) = (A + B)+ C.
3. A + 0 = 0 +A = A, where 0 is a null matrix of appropriate order.

11.4.3 : Subtraction of matrices:
If two matrices A and B are of the same order then the subtraction can be performed, the subtraction denoted by A – B is

the matrix of the same order which is obtained by subtracting the corresponding elements of B from the elements of A.

Definition 11.4.2:<u>Subtraction of matrices</u>: If $A = \left[a_{ij}\right]_{m \times n}$ and $B = \left[b_{ij}\right]_{m \times n}$ are two matrices then, $A - B = \left[a_{ij} - b_{ij}\right]_{m \times n}$.

Note that, matrices of different orders can not be sutracted.

Examples:

- If $A = \begin{bmatrix} 4 & 5 \\ 1 & 9 \end{bmatrix}$ and B = $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then A – B = $\begin{bmatrix} 4-1 & 5-2 \\ 1-3 & 9-4 \end{bmatrix}$ = $\begin{bmatrix} 3 & 3 \\ -2 & 5 \end{bmatrix}$

- If $C = \begin{bmatrix} 1 & 5 & 2 \\ 5 & 2 & 9 \\ 2 & 9 & 8 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 7 & 3 \\ 1 & 3 & 8 \end{bmatrix}$, then

  C – D = $\begin{bmatrix} 1-2 & 5-0 & 2-1 \\ 5-0 & 2-7 & 9-3 \\ 2-1 & 9-3 & 8-8 \end{bmatrix}$ = $\begin{bmatrix} -1 & 5 & 1 \\ 5 & -5 & 6 \\ 1 & 6 & 0 \end{bmatrix}$.

- If $P = \begin{bmatrix} -1 & 3 & 2 & 1 \\ 6 & 4 & 0 & 5 \end{bmatrix}$ and $Q = \begin{bmatrix} 3 & 5 & -2 & 1 \\ 7 & 9 & 4 & 5 \end{bmatrix}$, then

$$P - Q = \begin{bmatrix} -1-3 & 3-5 & 2-(-2) & 1-1 \\ 6-7 & 4-9 & 0-4 & 5-5 \end{bmatrix} = \begin{bmatrix} -4 & -2 & 4 & 0 \\ -1 & -5 & -4 & 0 \end{bmatrix}.$$

11.4.4 : Scalar Multiplication:

Definition 11.4. 3: <u>Scalar Multiplication</u>: If $A = \left[a_{ij}\right]_{m \times n}$ is any matrix and k is a scalar i.e. a real number then ka , the scalar multiple of A is defined as $kA = \left[ka_{ij}\right]_{m \times n}$.

Examples:

- If $A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ and k = $\frac{1}{3}$ then kA = $\begin{bmatrix} 2/3 & 2/3 \\ 2/3 & 2/3 \end{bmatrix}$.

- If $A = \begin{bmatrix} 1 & 5 & 2 \\ 5 & 2 & 9 \\ 2 & 9 & 8 \end{bmatrix}$ and k = 2, then kA $= \begin{bmatrix} 2 & 10 & 4 \\ 10 & 4 & 18 \\ 6 & 18 & 16 \end{bmatrix}$.

- If $Q = \begin{bmatrix} 3 & 5 & -2 & 1 \\ 7 & 9 & 4 & 5 \end{bmatrix}$ and k = 5, then $kQ = \begin{bmatrix} 15 & 25 & -10 & 5 \\ 35 & 45 & 20 & 25 \end{bmatrix}$.

Properties of scalar multiplication:

1. Scalar multiplication is distributive over addition and subtraction of matrices i.e. k (A ± B) = k A ± k B.
2. Addition an d subtraction of scalars is distributive over scalar multiplication i.e. ($k_1 \pm k_2$) A = k_1A ± k_2 A .
3. If k = –1, then kA = –A where – A $= -\left[a_{ij}\right]_{m \times n} = \left[-a_{ij}\right]_{m \times n}$.
4. If k = 0 then kA = 0, where 0 is a null matrix of appropriate order.

11.4.5 : Multiplication of Matrices: If two matrices A and B are such that the number of columns of A is equal to the number of rows of B then their multiplication denoted by AB can be performed. If the order of A is m × n and the order of B is n × p then the product matrix AB is of order m × p.

Definition 11.4.4: Multiplication of Matrices: If $A = \left[a_{ij}\right]_{m \times n}$ and $B = \left[b_{ij}\right]_{n \times p}$ are two matrices then,

$$AB = \left[a_{ij}\right]_{m \times n} \times \left[b_{ij}\right]_{n \times p}$$

$$= \left[\sum_{k=1}^{n} (a_{ik} \times b_{kj})\right]_{m \times p}$$

Examples:

- If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}_{2 \times 2}$ and $B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}_{2 \times 2}$, then

$$AB = \begin{bmatrix} 1\times1+1\times(-1) & -1\times1+1\times1 \\ 1\times1+1\times(-1) & -1\times1+1\times1 \end{bmatrix}_{2\times2} = \begin{bmatrix} 1-1 & -1+1 \\ 1-1 & -1+1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

- If $A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}_{2 \times 2}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2 \times 2}$, then

$$AB = \begin{bmatrix} 2\times1+2\times3 & 2\times2+2\times4 \\ 2\times1+2\times3 & 2\times2+2\times4 \end{bmatrix}_{2\times2} = \begin{bmatrix} 8 & 12 \\ 8 & 12 \end{bmatrix}$$

- If $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}_{2 \times 3}$ and $B = \begin{bmatrix} 7 & 1 & 5 & 2 \\ 3 & 8 & 4 & 7 \\ 6 & 3 & 9 & 5 \end{bmatrix}_{3 \times 4}$, then their product AB is a

matrix of order 2×4 and,

A B =

$$= \begin{bmatrix} 1\times7+3\times3+5\times6 & 1\times1+3\times8+5\times3 & 1\times5+3\times4+5\times9 & 1\times2+3\times7+5\times5 \\ 2\times7+4\times3+6\times6 & 2\times1+4\times8+6\times3 & 2\times5+4\times4+6\times9 & 2\times2+4\times7+6\times5 \end{bmatrix}_{2\times4}$$

$$= \begin{bmatrix} 46 & 40 & 62 & 48 \\ 62 & 52 & 80 & 62 \end{bmatrix}.$$

Properties of Multiplication of Matrices: If A,B,C are matrices such that their multiplications are possible, 0 denotes null matrix of appropriate order and I denotes identity matrix of appropriate order, then

1. Multiplication of matrices is not commutative. i.e. A B $\neq$ B A, for all matrices A and B.
2. Multiplication of matrices is associative, i.e. A (B C) = (A B) C.
3. A + 0 = 0 +A = A, where 0 is a null matrix of appropriate order
4. Multiplication of matrices is distributive over addition and subtraction of matrices i.e. A (B $\pm$ C) = A B $\pm$ A C.
   and (B $\pm$ C) A = B A $\pm$ C A.
5. Identity matrix is multiplicative identity i.e. I A = A I = A .
6. 0 A = A 0 = 0, where 0 is a null matrix of appropriate order.
7. It is possible that A B = 0, but A and B are such that A $\neq$0 as well as B $\neq$0.

***Self Test II*:**

11.5 : Determinant:

If A is any square matrix of real numbers, then the determinant of A is a certain real number assigned to the matrix A. It is denoted by | A | or det(A). The formula to find the determinant changes according to the order of the matrix.

Definition 11.5.1: Determinant of a Matrix of order 2×2 :

If A = $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is a 2×2 matrix, then its determinant is defined as ,

$$| A | = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\, a_{22} - a_{21}\, a_{12} \ .$$

Example:

- If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2 \times 2}$ then its determinant is,

  $| A | = 1 \times 4 - 3 \times 2 = 4 - 6 = -2.$

Definition 11.5.2: Determinant of a Matrix of order 3×3 :

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a 3×3 matrix, then its determinant is defined as ,

$$| A | = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}\,(a_{22}\, a_{33} - a_{32}\ \ a_{33}) - a_{12}\,(a_{21}\, a_{33} - a_{31}\ \ a_{33}) + a_{33}\,(a_{21}\, a_{32} - a_{31}\, a_{22}) \ .$$

Example:

- If $A = \begin{bmatrix} 1 & 5 & 2 \\ 5 & 2 & 9 \\ 2 & 9 & 8 \end{bmatrix}$ is a 3×3 matrix then ,its determinant is,

$|A| = 1 \times (2\times8 - 9\times9) - 5 \times(5\times8 - 2\times9) +2 \times(5\times 9 - 2\times 2)$

$= 1 \times (16 - 81) - 5 \times(40 - 18) +2 \times(45 - 4)$

$= 1 \times (- 65) - 5 \times(22) +2 \times(41)$

$= - 65 - 110 + 82$

$= - 93.$

Definition 11.5.3: Singular Matrix: A square matrix A is said to be singular matrix if its determinant is equal to zero i.e. $|A | = 0$. Otherwise it is called as non singular matrix.

Examples:

- If $A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}_{2 \times 2}$ then $| A | = 2 \times 2 - 2 \times 2 = 4 - 4 = 0.$

  $\therefore$ A is a singular matrix.

- and $B = \begin{bmatrix} 4 & 5 \\ 3 & 6 \end{bmatrix}_{2 \times 2}$, then $| B | = 4 \times 6 - 3 \times 5 = 24 - 15 = 9 \neq 0.$

$\therefore$ B is a non singular matrix.

<u>Properties of determinants:</u>

(i) If the rows and the columns of a matrix are interchanged then the value of the determinant remains the same. Hence for every square matrix and its transpose the value of the determinant is same. i.e. for any square matrix A , we have $|A| = |A^t|$.

(ii) If any two rows (or columns) of a matrix are interchanged then the value of the determinant changes in sign only.

(iii) If any two rows (or columns) of a matrix are identical then the value of the determinant is zero.

(iv) If every element of any one row (or column) of a matrix is multiplied by a constant k, then value of the determinant is multiplied by k.

<u>Self Test</u> **<u>III</u>:**

11.6: Inverse of a matrix:

Definition 11.6.1: <u>Inverse of a Matrix:</u> If two square matrices A and B are such that AB = BA = I, where I denotes identity matrix of the same order, then B is known as inverse of A, and it is denoted by A^{-1}.If B is inverse of A then A is inverse of B. i.e. if $B = A^{-1}$ then $A = B^{-1}$.

Inverse of every square matrix does not exist. If A is a matrix such that its inverse exists then it is called as invertible matrix.

It is known that a matrix is invertible if and only if its determinant is nonzero i.e. if and only if it is a non singular matrix.

It is possible to find inverse of an invertible matrix using various methods. We will now study one of the methods to find inverse of a matrix which is using adjoint of the matrix. To understand what is adjoint we need to understand few more terms such as minor and cofactor.

Definition 11.6.2: <u>Minor of an element:</u> If $A = \left[a_{ij}\right]_{n\times n}$ is any square matrix, then the minor of the (i, j)th element a_{ij} is the determinant of the submatrix obtained by deleting i^{th} row and j^{th} column from A. It is denoted by m_{ij}.

Example:

- If $A = \begin{bmatrix} 1 & 1 & -4 \\ 2 & 6 & 2 \\ 1 & 0 & -1 \end{bmatrix}$ is a square matrix, then we can find minor m_{ij} of every

element a_{ij} of A . By the above definition

m_{ij} =determinant of the submatrix obtained by deleting i^{th} row and j^{th} column of A.

$\therefore$ m_{11} =determinant of the submatrix obtained by deleting 1^{st} row and 1^{st} column of A

$\therefore$ $m_{11} = \begin{vmatrix} 6 & 2 \\ 0 & -1 \end{vmatrix} = 6 \times (-1) - 0 \times 2 = -6 - 0 = -6.$

Similarly $m_{12} = \begin{vmatrix} 2 & 2 \\ 1 & -1 \end{vmatrix} = 2 \times (-1) - 1 \times 2 = -2 - 2 = -4.$

$m_{13} = \begin{vmatrix} 2 & 6 \\ 1 & 0 \end{vmatrix} = 2 \times 0 - 1 \times 6 = 0 - 6 = -6.$

$m_{21} = \begin{vmatrix} 1 & -4 \\ 0 & -1 \end{vmatrix} = 1 \times (-1) - 0 \times (-4) = -1 - 0 = -1.$

$m_{22} = \begin{vmatrix} 1 & -4 \\ 1 & -1 \end{vmatrix} = 1 \times (-1) - 1 \times (-4) = -1 + 4 = 3.$

$m_{23} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1 \times 0 - 1 \times 1 = 0 - 1 = -1.$

$m_{31} = \begin{vmatrix} 1 & -4 \\ 6 & 2 \end{vmatrix} = 1 \times 2 - 6 \times (-4) = 2 + 24 = 26.$

$m_{32} = \begin{vmatrix} 1 & -4 \\ 2 & 2 \end{vmatrix} = 1 \times 2 - 2 \times (-4) = 2 + 8 = 10.$

$m_{33} = \begin{vmatrix} 1 & 1 \\ 2 & 6 \end{vmatrix} = 1 \times 6 - 1 \times 2 = 6 - 2 = 4.$

Definition 11.6.3: <u>Cofactor of an element:</u> If $A = \left[a_{ij}\right]_{n \times n}$ is any square matrix, and the minor of the (i, j)th element a_{ij} is m_{ij}, then the cofactor of the element a_{ij} is denoted by c_{ij} is defined as $c_{ij} = (-1)^{i+j} \times m_{ij}$.

Example:

- For a square matrix $A = \begin{bmatrix} 1 & 1 & -4 \\ 2 & 6 & 2 \\ 1 & 0 & -1 \end{bmatrix}$, we have minor m_{ij} of every element a_{ij} of A , as computed above. Using these minors we can find the corresponding cofactors $c_{ij} = (-1)^{i+j} \times m_{ij}$,as below,

As $m_{11} = -6 \quad \therefore c_{11} = (-1)^{1+1} \times m_{11} = (-1)^2 \times (-6) = 1 \times (-6) = -6$.

Similarly $m_{12} = -4. \quad \therefore c_{12} = (-1)^{1+2} \times m_{12} = (-1)^3 \times (-4) = (-1) \times (-4) = 4$.

$m_{13} = -6. \qquad \therefore c_{13} = (-1)^{1+3} \times m_{13} = (-1)^4 \times (-6) = 1 \times (-6) = -6$.

$m_{21} = -1. \qquad \therefore c_{21} = (-1)^{2+1} \times m_{21} = (-1)^3 \times (-1) = (-1) \times (-1) = 1$.

$m_{22} = 3. \qquad \therefore c_{22} = (-1)^{2+2} \times m_{22} = (-1)^4 \times 3 = 1 \times 3 = 3$.

$m_{23} = -1. \qquad \therefore c_{23} = (-1)^{2+3} \times m_{23} = (-1)^5 \times (-1) = (-1) \times (-1) = 1$.

$m_{31} = 26. \qquad \therefore c_{31} = (-1)^{3+1} \times m_{31} = (-1)^4 \times 26 = 1 \times 26 = 26$.

$m_{32} = 10. \qquad \therefore c_{32} = (-1)^{3+2} \times m_{32} = (-1)^5 \times 10 = (-1) \times 10 = -10$.

$m_{33} = 4. \qquad \therefore c_{33} = (-1)^{3+3} \times m_{33} = (-1)^6 \times 4 = 1 \times 4 = 4$.

Definition 11.6.4: <u>Cofactor matrix</u>: If $A = \left[a_{ij}\right]_{n \times n}$ is any square matrix then cofactor matrix of A is the matrix $C = \left[c_{ij}\right]_{n \times n}$ in which the (i, j)th element c_{ij} is the cofactor of the element a_{ij} of A.

Example:

- For a square matrix $A = \begin{bmatrix} 1 & 1 & -4 \\ 2 & 6 & 2 \\ 1 & 0 & -1 \end{bmatrix}$, we have cofactor c_{ij} of every element a_{ij} of A, as computed above. Using these cofactors we can write the cofactor matrix C of A as below,

$$C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} = \begin{bmatrix} -6 & 4 & -6 \\ 1 & 3 & 1 \\ 26 & -10 & 4 \end{bmatrix}.$$

Definition 11.6.5: <u>Adjoint of a Matrix:</u> Adjoint of a matrix A is the transpose of the matrix of cofactors of A. This is denoted by Adj(A).

Using these definitions and matrix algebra, some theorems can be proved, which give a formula to find inverse of any square nonsingular matrix.

This formula is stated as $A^{-1} = \frac{1}{|A|} \times Adj(A)$.

Examples:

- For a square matrix $A = \begin{bmatrix} 1 & 1 & -4 \\ 2 & 6 & 2 \\ 1 & 0 & -1 \end{bmatrix}$, we have cofactor matrix C of A as

C = $\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} = \begin{bmatrix} -6 & 4 & -6 \\ 1 & 3 & 1 \\ 26 & -10 & 4 \end{bmatrix}$.

$\therefore$ Adj(A) = C^t = $\begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix} = \begin{bmatrix} -6 & 1 & 26 \\ 4 & 3 & -10 \\ -6 & 1 & 4 \end{bmatrix}$.

The inverse of A is computed by $A^{-1} = \frac{1}{|A|} \times Adj(A)$.

Now |A| = $\begin{vmatrix} 1 & 1 & -4 \\ 2 & 6 & 2 \\ 1 & 0 & -1 \end{vmatrix}$

= $1\times[6 \times (-1) - 0 \times 2\,] - 1 \times[2 \times (-1) - 1 \times 2] + (-4) \times [2 \times 0 - 1 \times 6]$

= $[1\times(-6)\,] - [1 \times (-4)] + [(-4) \times(-6)\,]$

= $-6 + 4 + 24$

= $22 \neq 0$. $\therefore$ A is invertible matrix .

$$\therefore A^{-1} = \frac{1}{22} \times \begin{bmatrix} -6 & 1 & 26 \\ 4 & 3 & -10 \\ -6 & 1 & 4 \end{bmatrix} = \begin{bmatrix} -\frac{6}{22} & \frac{1}{22} & \frac{26}{22} \\ \frac{4}{22} & \frac{3}{22} & \frac{-10}{22} \\ \frac{-6}{22} & \frac{1}{22} & \frac{4}{22} \end{bmatrix}$$

- If B = $\begin{bmatrix} 4 & 5 \\ 3 & 6 \end{bmatrix}_{2 \times 2}$, then | B | = $4 \times 6 - 3 \times 5 = 24 - 15 = 9 \neq 0$.

$\therefore$ B is a non singular matrix. $\therefore$ B is invertible matrix, hence by definition of minor we have,

m_{11} =determinant of the submatrix obtained by deleting 1st row and 1st column of B $\therefore$ $m_{11} = |\ 6\ | = 6$.

And similarly $m_{12} = |\ 3\ | = 3$, $m_{21} = |\ 5\ | = 5$ and $m_{22} = |\ 4\ | = 4$.

Using these minors we can find the corresponding cofactors $c_{ij} = (-1)^{i+j} \times m_{ij}$,as $c_{11} = (-1)^{1+1} \times m_{11} = (-1)^2 \times 6 = 1 \times 6 = 6$.

$c_{12} = (-1)^{1+2} \times m_{12} = (-1)^3 \times 3 = (-1) \times 3 = -3$.

$c_{21} = (-1)^{2+1} \times m_{21} = (-1)^3 \times 5 = (-1) \times 5 = -5$.

and $c_{22} = (-1)^{2+2} \times m_{22} = (-1)^4 \times 4 = 1 \times 4 = 4$.

Hence cofactor matrix C $= \begin{bmatrix} 6 & -3 \\ -5 & 4 \end{bmatrix}$, $\therefore$ Adj(B) = $C^t = \begin{bmatrix} 6 & -5 \\ -3 & 4 \end{bmatrix}$.

By formula to find inverse, $B^{-1} = \frac{1}{|B|} \times Adj(B) = \frac{1}{9} \times \begin{bmatrix} 6 & -5 \\ -3 & 4 \end{bmatrix}$.

Self Test IV:

11.7:Summary for Unit 11:

In this unit learners studied the following topics in details:

1. The concept of concept of matrix.

2. Different types of matrices such as, equal matrices, identity matrix, null matrix, diagonal matrices and singular matrices etc.

3. Operations on matrices such addition, subtraction and scalar multiplication and multiplication of matrices.

4. Concept of determinant. Evaluation of determinant of a matrix.

5. Concept of inverse of a matrix and adjoint method to find inverse of a matrix.