Unit 7 : Mathematical Logic

7.0 Unit Objectives:

By the end of this Unit, learners should be able to:

- Define statements in accordance with mathematical logic.
- Find the truth value of a statement.
- Construct the truth tables for different propositional forms.
- Explain various logical connectives and compound statement.
- Define logical equivalence.
- Prove the logical equivalence of statement patterns.
- Understand tautology and a contradiction.
- Find the converse, inverse and contrapositive of a conditional statement.

7.1 Unit Introduction:

In this unit we will discuss about one of the oldest branches of mathematics which is known as Logic. Logic is considered as the science of reasoning. The earliest use of mathematics and geometry in relation to logic and philosophy goes back to the ancient Greeks such as Euclid, Plato, and Aristotle.

The field of logic ranges from core topics such as the study of fallacies and paradoxes to specialized analysis of reasoning using probability. Traditionally, logic is studied as a branch of philosophy. And in modern era we know that the development of formal logic and its implementation in computing machinery is the foundation of computer science.

7.2: Statements in Mathematical Logic :

While studying Mathematical logic we deal with a discrete mathematical object which is a Statement or proposition.

Definition: 7.2.1: **Statement :** A statement is defined as a declarative sentence, which is either true or false, but not both at a time.

It is also known as a proposition or an atomic statement.

Examples:

Consider the following sentences,

- "The earth rotates around the sun."

This sentence is declarative as it is giving us some information. It is a true statement. Hence it is a logical Statement or a proposition.

- "It is not raining today. "

This sentence is also giving some information. It is either true or false, but of course, not both at any particular time. Hence it is a Statement.

- "Give me that book."

This sentence is not giving any information. Also we can not decide whether it is true or false .Hence it is not a Statement.

- "Are you interested in Cricket? "

This question is not giving any information. Also we can not decide whether it is true or false .Hence it is not a statement.

- "The positive divisors of 10 are 1, 2, 5 and 10 only."

This sentence is declarative as it is giving some information. It is a true statement. Hence it is a logical Statement or a proposition.

- " $4 + 15 > 20$."

This sentence is giving information about the addition of the numbers 4 and 15. It is false as $4 + 15 = 19 < 20$. Hence it is a Statement.

- "19 is a prime number."

This sentence is giving information about the number 19. It is a true statement, as 19 is in fact a prime number. Hence it is a Statement.

- "How nice!"

This sentence is not giving information about anything. Hence it is not a Statement.

- "She is tall."

This is not a statement because, it is not giving any particular information as who she is we do not know.

So we know that Imperative sentences, Exclamatory sentences and Interrogative sentences are not statements in Logic. The fundamental identities are statements in Logic.

<u>Definition: 7.2.2:</u> **Truth value of a statement:** The truth-ness or falsity of the statement is known as the truth value of the statement.

If the statement is true, we say that its truth value is true and denote it by the letter 'T' or by the number "1". If the statement is false, we say that its truth value is False and it is denoted by the letter 'F' or by the number '0' .

Examples:

Consider the following sentences,

- "The earth rotates around the sun".

This sentence is true. Hence it is a logical Statement which has truth value "T".

- "The positive divisors of 10 are 1, 2, 5 and 10. only. "

This sentence is true. So it has truth value "T".

- "$4 + 15 > 20$."

This sentence is false. So it has truth value "F".

- "18 is a prime number."

This sentence is false. So it has truth value "F".

❖ ***Self Test I*:**

Select the correct alternative from the given alternatives.

1. Which of the following is a logical statement?

(a) Mumbai is in India. **(b)** What are you doing?

(c) He is a kind person. **(d)** $x + 9 = 12$.

2. Which of the following is not a logical statement?

(a) Mumbai is in India. **(b)** Put that pen on the table.

(c) It is raining. **(d)** $8 + 9 = 12$.

3. Which of the following is a logical statement?

(a) If x is tall then y is handsome. **(b)** Please, be seated.

(c) Kapil Deo is a great cricketer. **(d)** $x + 9y = 12$.

4. Which of the following is not a logical statement?

(a) Mumbai is in India and $19 \geq 12$. **(b)** How are you?.

(c) It is raining or it is not raining. **(d)** 8 is a square of 4.

5. Which of the following is a true logical statement?

(a) Mumbai is in England. **(b)** Are you going home?.

(c) The moon rotates around the earth. **(d)** 8 is a square of 4.

6. Which of the following is a false logical statement?

(a) London is in England. **(b)** Are you going home?.

(c) The earth rotates around the sun. **(d)** 8 is the cube of 4.

7. What is the truth value of the statement,"1 is a prime integer." ?

(a) T **(b)** F **(c)** Both T and F **(d)** Neither T nor F.

8. What is the truth value of the statement, "1 is an odd integer." ?

(a) T **(b)** F **(c)** Both T and F **(d)** Neither T nor F.

9. What is the truth value of the statement, "x is a prime integer." ?

(a) T **(b)** F **(c)** Both T and F **(d)** it is not a statement.

10. What is the truth value of the statement, " Tiger is a wild animal." ?

(a) T **(b)** F **(c)** Both T and F **(d)** Neither T nor F.

7.2.1 :Logical connectives:

In our day-to-day language, we form new sentences with two or more sentences. Similarly in mathematical logic we use some words to form new statements. Such words are called logical connectives. Examples of logical connectives are words such as ‘and’, ‘or’ and ‘not’. Also the phrases such as if - then and if and only if are examples of logical connectives.

Types of logical statements:

Definition :7.3.3: **Simple statement** : A logical statement in which no connective is used is called a “simple statement “or a proposition or an atomic statement . Generally small case letters are use to denote simple statement.

Examples :

p: London is in England.
q: 2 + 7 = 10.
r: 21 is a multiple of 9.
s: 11 > 121

Definition :7.3.4: **Compound statement** :

If a statement is formed by joining 2 or more simple statements using logical connectives then it is called a “compound statement” or “a propositional form.

Examples:

p: Mumbai is in England and it is raining.
q: 2 + 7 = 10 or 1 is a prime number.
r: If 21 is a multiple of 9 then 2+ 2= 4.
s: It is not true that , 11 > 121.

7.2.2: Types of Compound statements:

The important compound statements are negation, conjunction, disjunction, conditional and biconditional statements. The truth values of these connectives are defined by means of the truth tables.

A truth table is a computational device by which we can determine the truth values of compound statements. We tabulate all possible values of the simple statements in it. If the compound statement contains n variables i.e. n simple statements then the truth table contains 2^n rows.

1. Negation : Negation is the compound statement in which the word “not" is the connective used. We always use negations of the sentences to indicate exactly opposite meaning.

If ‘p’ is a simple statement, then the statement not p is known as the **negation of p**. It is denoted symbolically as ~p.
Clearly, if p is true statement then its negation is false and if p is false then ~p is true statement. So the truth values of the negation are as per the following truth-table.

p	~ p
T	F
F	T

Examples:

- If p: Logic is easy.

  then its negation is the statement, ~p : Logic is not easy.
- If q: 2 is not a rational number .

  then its negation is , ~q : 2 is a rational number.
- If p: The earth rotates around the moon.

  then its negation is , ~p : The earth does not rotate around the moon.

Note that, while writing the negations we use the logical connective “not” and do not write the words of opposite meaning.

2. **<u>Conjunction</u>** :

Conjunction is the compound statement in which connective used is the word “and”. Suppose p and q are any two statements. The conjunction of ‘p’ and ‘q’ is defined as the statement “p and q“
It is denoted symbolically as $p \wedge q$.
Example : If p : Sunday is a holiday,
q : Every day I study for at least 4 hours,
then the conjunction of p and q is
$p \wedge q$: Sunday is a holiday and Every day I study for at least 4 hours.
In case of Conjunction, the statement $p \wedge q$ is true if both p as well as q are true statements else $p \wedge q$ is a false statement. Therefore the truth table of conjunction is:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

3. <u>Disjunction</u> :

Disjunction is the compound statement in which the connective "or" is used . Suppose p and q are two statements. Then the disjunction of 'p' and 'q' is the statement "p or q". It is denoted symbolically as $p \vee q$.

Example : If p : It is raining now. q : The air is fresh,
then the disjunction of p and q is

$p \vee q$: It is raining now or the air is fresh.

In case of Disjunction, the statement $p \vee q$ is true if either p or q is true and if both p and q are false the Disjunction is false.
Therefore the truth table of disjunction is:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

4. <u>Conditional Statement or Implication</u>:

Conditional Statement is a compound statement in which, we combine two statements by the phrase 'If - , then' or "implies that". This gives us a new statement. Suppose p and q are two statements. Then the compound statement

" If p, then q"

is called a conditional statement. We denote this conditional statement symbolically as $p \rightarrow q$.

The statement ‘p’ is known as the hypothesis or antecedent. The statement ‘q’ is known as the conclusion or consequent.

The conditional statement “if p then q” can be phrased in different ways as :

“p only if q “

or “ q, provided that p”

or “q if p”

or “p implies q”

or “ p is a sufficient condition for q”

or “q is a necessary condition for p”.

Examples:

- If two simple statements are,

p: The student has a quest for knowledge.

q: The student will take up new courses.

then the conditional statement is :“If the student has a quest for knowledge, then he will take up new courses”

- Suppose two simple statements are, p: 3 +5 = 9 and q: 5 is even number. Then the conditional statement is: “5 is even number if, 3 +5 = 9”.

A conditional statement is False only when hypothesis ‘p’ is true, but the conclusion ‘q’ is False. It is true for all remaining combinations of the truth values of p and q. Therefore the truth table of a conditional statement is:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

5. Biconditional Statement or Double Implication:

Biconditional Statement is a compound statement in which, we combine two statements by the phrase "If and only if". Suppose p and q are any two statements. Then the compound statement "p if and only if q" is called a biconditional statement. It is denoted by $p \leftrightarrow q$

Examples:

- If two simple statements are,

p: The student has a quest for knowledge.

q: The student will take up new courses.

Then the biconditional statement is: "The student has a quest for knowledge if and only if he will take up new courses"

- If two simple statements are, p: 3 +5 = 9 and q: 5 is even number. Then the conditional statement is: " 3 +5 = 9 if and only if 5 is even number".

The biconditional statement "p if and only if q" can be phrased in different words as "p is a necessary and sufficient condition for q". The bi-conditional statement $p \leftrightarrow q$, is true if both p and q are having the same truth values.

The truth table of the biconditional statement $p \leftrightarrow q$ is as follows:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Using these 5 connective different compound statement patterns can be formed. We can decide the truth values of such statement patterns by preparing their truth tables.

Examples:

- The truth table for the compound statement $p \wedge \sim q$ contains 4 columns which represent the statements p, q , $\sim q$ and $p \wedge \sim q$ respectively. The table is as follows:

p	q	$\sim q$	$p \wedge \sim q$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

- Similarly the truth table for the compound statement $(p \vee q) \wedge \sim p$ is as follows:

p	q	$p \vee q$	$\sim p$	$(p \vee q) \wedge \sim p$
T	T	T	F	F
T	F	T	F	F
F	T	T	T	T
F	F	F	T	F

- The truth table for the statement pattern $(\sim p \vee \sim q) \vee p$ is as follows:

p	q	$\sim p$	$\sim q$	$\sim p \vee \sim q$	$(\sim p \vee \sim q) \vee p$
T	T	F	F	F	T
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	T	T

- The truth table for the compound statement $\sim(p \wedge q) \vee \sim(q \vee r)$ is as follows:

As this statement pattern contains 3 simple variables, there are 2^3 combinations of truth values. Hence there are $2^3 = 8$ rows in this truth table.

p	q	r	$p \wedge q$	$q \vee r$	$\sim(p \wedge q)$	$\sim(q \vee r)$	$\sim(p \wedge q) \vee \sim(q \vee r)$
T	T	T	T	T	F	F	F
T	T	F	T	T	F	F	F
T	F	T	F	T	T	F	T
T	F	F	F	F	T	T	T
F	T	T	F	T	T	F	T
F	T	F	F	T	T	F	T
F	F	T	F	T	T	F	T
F	F	F	F	F	T	T	T

❖ ***Self Test II*:**

Exercise 1: Select the correct alternative from the given alternatives.

1. Which of the following is a simple statement?

(a) Mumbai is in India and Paris is in England.

(b) If roses are red then violets are blue.

(c) It is not raining. **(d)** $9 \geq 12$.

2. Which of the following is a compound statement?

(a) $9 \geq 12$. **(b)** Paris is in England.

(c) It is raining or it is cold. **(d)** Roses are red.

3. Which of the following words is a connective in logic?

(a) yes **(b)** in fact **(c)** then **(d)** then.

4. Which of the following words is not a connective in logic?

(a) and **(b)** or **(c)** then **(d)** not.

5. Which of the following is an example of a negation?

(a) Mumbai is in India and Paris is in England.

(b) If roses are red then violets are blue.

(c) It is not raining. **(d)** $9 \geq 12$.

6. Which of the following is an example of a conjunction?

(a) $9 \geq 12$**.** **(b)** It is raining or it is cold.

(c) Paris is in England and London is in India.

(d) If roses are red then violets are blue.

7. Which of the following is an example of a implication?

(a) Mumbai is in India and Paris is in England.

(b) If roses are red then violets are blue.

(c) It is not raining. **(d)** $9 \geq 12$.

8. Which of the following is an example of a disjunction?

(a) $9 \geq 12$ if and only if $7 < 16$**.** **(b)** It is raining or it is cold.

(c) Paris is in England and London is in India.

(d) If roses are red then violets are blue.

9. Which of the following is an example of a biconditional statement?

(a) $9 \geq 12$ if and only if $7 < 16$**.** **(b)** It is raining or it is cold.

(c) Paris is in England and London is in India.

(d) If roses are red then violets are blue.

10. How many rows are their in the truth table, if the compound statement pattern contains 5 simple statements?

(a) 2^4 **(b)** 4 **(c)** 8 **(d)** 2^5.

Exercise 2: **Write the truth table of each statement pattern given below:**

1. $\sim (p \leftrightarrow q)$

2. $\sim p \vee \sim q$

3. $(p \wedge q) \rightarrow (p \vee \sim q)$

4. $(p \rightarrow q) \rightarrow r$

5. $[\sim (p \wedge q)] \leftrightarrow (q \vee r)$

7.3 : Logical Equivalence :

In many cases two different compound statements which are formed by the same simple statements (but in which different connectives appear), have the same truth values. Such compound statements are logically equivalent statements. In such cases, we can replace one compound statement by another. For example, you may come across a complex logical condition in a computer program. You can replace it by another logically equivalent, but simpler condition. We define this concept formally as follows.

Definition 7.3.1: <u>Logically Equivalent statements:</u>

Suppose that P and Q are two compound statements made up of the same atomic statements p_1, p_2, **…,** p_n. We say that P and Q are logically equivalent if for every combination of the truth values of the atomic statements p_1, p_2, **…,** p_n; the truth values of P and Q are identical.

If P and Q are logically equivalent, we denote this by writing, $P \equiv Q$.

To verify whether P and Q are logically equivalent, we construct the truth table for P and Q. If the columns which give truth values of P and Q are identical, then P and Q are logically equivalent. Otherwise they are not logically equivalent.

Examples:

- Verify whether $p \to q \equiv {\sim}p \vee q$

proof : We construct the truth table as below

1	2	3	4	5
p	q	$p \to q$	${\sim}p$	$({\sim}p) \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

From this table we observe that the column corresponding to $p \to q$ i.e. 3rd column in the table has same truth values as that of the column corresponding to ${\sim}p \vee q$ i.e. 5th column in the table. Hence we conclude that $p \to q \equiv {\sim}p \vee q$.

- Verify whether $\sim(p \to q) \equiv p \wedge (\sim q)$

proof : We construct the truth table as below

1	2	3	4	5	6
p	q	$p \to q$	$\sim(p \to q)$	$\sim q$	$p \wedge (\sim q)$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	F	T	F

From this table we observe that the column corresponding to $\sim(p \to q)$ i.e. 3^{rd} column in the table has same truth values as that of the column corresponding to $p \wedge (\sim q)$ i.e. 6^{th} column in the table. Hence we conclude that

$\sim(p \to q) \equiv p \wedge (\sim q)$.

7.3.1: Logical identities

There are many important logical equivalences called logical identities, few of them are as follows, where p, q and r denote any statements:

1. Double negation law: $\sim(\sim p) \equiv p$
2. Demorgan laws: $\sim(p \wedge q) \equiv \sim p \vee \sim q$
   $\sim(p \vee q) \equiv \sim p \wedge \sim q$
3. Commutative laws: $p \wedge q \equiv q \wedge p$
   $p \vee q \equiv q \vee p$
4. Associative laws: $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
   $(p \vee q) \vee r \equiv p \vee (q \vee r)$
5. Distributive laws: $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
   $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
6. $p \to q \equiv \sim q \to \sim p$
7. $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

We can prove these equivalences by constructing their truth tables.

7.4 : Tautology and contradiction :

A compound statement which is always true is called a tautology and a compound statement which is always false is called a contradiction. A statement which is neither a tautology nor a contradiction is called a contingent statement. Note that If p is a tautology, then ~p is a contradiction and if p is a contradiction then ~p is a tautology.

Definition 7.4.1: <u>Tautology</u> : If P is a compound statement made up of the atomic statements $p_1, p_2, \ldots, p_n$ and connectives; and if for every combination of the truth values of the atomic statements $p_1, p_2, \ldots, p_n$; the truth value of P is True i.e T, then P is a tautology.

Example:

- Verify whether $(p \wedge (p \rightarrow q)) \rightarrow q$ is a tautology or not.

p	q	$p \rightarrow q$	$p \wedge (p \rightarrow q)$	$(p \wedge (p \rightarrow q)) \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

As all values in the last column are T, the compound statement $(p \wedge (p \rightarrow q)) \rightarrow q$ is a tautology

Definition 7.4.2: <u>Contradiction:</u> If P is a compound statement made up of the atomic statements $p_1, p_2, \ldots, p_n$ and connectives; and If for every combination of the truth values of the atomic statements $p_1, p_2, \ldots, p_n$; the truth value of P is False i.e., F then the compound statement P is a contradiction .

Example:

- Verify whether $(p \wedge q) \wedge \sim q$ is a contradiction or not.

p	q	$p \wedge q$	~q	$(p \wedge q) \wedge \sim q$
T	T	T	F	F
T	F	F	T	F
F	T	F	F	F
F	F	F	T	F

As all values in the last column are F, $(p \wedge q) \wedge \sim q$ is a Contradiction.

7.5: Converse, inverse and contrapositive:

One important concept in mathematical logic is of converse, inverse and contrapositive of a conditional statement. Using these concepts many results in mathematics are proved easily.

Definition 7.5.1: **<u>Converse of a conditional statement:</u>** Converse of a conditional statement $p \to q$, is the conditional statement $q \to p$.

Definition 7.5.1: **<u>Inverse of a conditional statement:</u>** Converse of a conditional statement $p \to q$, is the conditional statement $\sim p \to \sim q$.

Definition 7.5.1: **<u>Contrapositive of a conditional statement:</u>** Converse of a conditional statement $p \to q$, is the conditional statement $\sim q \to \sim p$.

Examples:

- Consider a conditional statement, “If two triangles are congruent then their sides are equal.” This is in $p \to q$ form where p: two triangles are congruent, and q: their sides are equal.

For this statement we have,

<u>Converse:</u> $q \to p$: If their sides are equal then two triangles are congruent.

<u>Inverse :</u> $\sim p \to \sim q$: If two triangles are not congruent then their sides are not equal.

<u>Contrapositive :</u> $\sim q \to \sim p$: If their sides are not equal then two triangles are not congruent.

- If a conditional statement in $p \rightarrow q$ form is: "If a function is bijection then the inverse function exists. " For this statement we have,

<u>Converse:</u> $q \rightarrow p$: If the inverse function exists then a function is bijection.

<u>Inverse :</u> $\sim p \rightarrow \sim q$: If a function is not a bijection then the inverse function does not exist.

<u>Contrapositive</u> : $\sim q \rightarrow \sim p$: If the inverse function does not exist then a function is not bijection.

- $p \rightarrow q$: " If Sachin receives a scholarship then he will study further."
  For this statement we have,

<u>Converse:</u> $q \rightarrow p$: If Sachin will study further then he receives a scholarship.

<u>Inverse :</u> $\sim p \rightarrow \sim q$: " If Sachin does not receive a scholarship then he will not study further.

<u>Contrapositive</u> : $\sim q \rightarrow \sim p$: If Sachin will not study further then he does not receive a scholarship.

❖ ***Self Test III***:

Exercise 1: Write the truth table of each of the following and determine whether it is a tautology or contradiction or a contingent statement.

(i) $p \wedge \sim q$

(ii) $(p \wedge q) \rightarrow q$

(iii) $(p \rightarrow q) \wedge (p \wedge \sim q)$

(iv) $p \wedge \sim p$

(v) $p \vee \sim p$

(vi) $[(p \leftrightarrow q) \wedge q] \rightarrow p$

(vii) $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$

(viii) $(p \wedge q) \wedge \sim (p \vee q)$

(ix) $\sim p \wedge (p \wedge q)$

(x) $(p \wedge q) \vee (p \wedge r)$

Exercise 2: Using truth table determine whether following statement patterns are logically equivalent.

(i) $p \wedge q$ and $p \vee q$

(ii) $\sim(p \vee q)$ and $\sim p \wedge \sim q$

- **(iii)** $\sim(p \wedge q)$ and $\sim p \vee \sim q$
- **(iv)** $\sim(p \wedge q)$ and $\sim p \wedge \sim q$
- **(v)** $p \wedge (q \vee r)$ and $(p \wedge q) \vee (p \wedge r)$
- **(vi)** $p \to q$ and $\sim q \to \sim p$
- **(vii)** $p \vee q$ and $\sim p \vee \sim q$
- **(viii)** $\sim(p \wedge q)$ and $\sim p \wedge \sim q$
- **(ix)** p and $p \wedge (p \vee q)$
- **(x)** $p \vee q$ and $q \vee p$

Exercise 3 : Write the converse, inverse and contrapositive of each of the following conditional statements.

- **(i)** If interest rates are low then the economy is good.
- **(ii)** If you are good in logic then you are good in mathematics.
- **(iii)** A number is divisible by 2 implies that it is an even number.
- **(iv)** A sufficient condition for Meena to visit Agra is that she goes to Tajmahal.
- **(v)** Violets are blue if roses are red.

7.6: Summary for unit 7:

In this unit learners studied the following topics in details:

1. A logical statement, and the definition of its truth value.
2. Simple and compound statements.
3. Important logical connectives which are negation, conjunction, disjunction, conditional and biconditional, with their truth tables.
4. Logical equivalence and how to prove it.
5. A tautology, a contradiction and a contingent statement.
6. Converse, inverse and contrapositive of a conditional statement.