Unit 3: Mathematical Induction:

3.0 Unit Objectives:

By the end of this Unit, learners will be able to:

- Understand the first principle of mathematical induction.
- Write proofs of statements using the method of mathematical induction.

3.1 Unit Introduction:

Mathematical induction is a method of proving a particular class of conjectures. A conjecture in mathematics is a statement which is not proved. Induction is generally used to establish that, a given statement is true for all natural numbers. The method can be extended to prove statements about more general structures; this generalization, known as structural induction or strong induction is used in Mathematical logic and computer science. In fact, mathematical induction is a form of deductive reasoning. The earliest traces of mathematical induction can be found in Euclid's proof that the set of all primes is an infinite set.

A method of proving conjectures is needed because, in mathematics "disproof by counterexample" always works but "proof by example" seldom works. But if the conjecture is a statement about a small finite set then we can prove the conjecture by showing that for each member of the collection it is true.

Example: Suppose we want to prove the following statement

"If a natural number from 1 to 13 is divisible by 6, then it is also divisible by 3."

One proof can be written as:

From 1 to 13, 6 is divisible by 6, it is also divisible by 3.

12 is divisible by 6 and it is also divisible by 3.

1,2,3,4,5,7,8,9,10,11,and 13 are not divisible by 6.

So we can say that above statement is proved, as it is checked for the two numbers in question.

But if we want to prove any statement for all natural numbers then this method is not useful as it is impossible to verify the statement for all numbers. So to prove a conjecture is true, we need some more formal methods of proof. One of these methods is of mathematical induction.

Principle of Mathematical Induction can be stated as:

"Show that something works for the first time.
Assume that it works for this time, and show that it will work for the next time.
Conclusion, it works all the time."

3.2 The first principle of mathematical induction:

The simplest and most common form of mathematical induction is the first principle of mathematical induction.

Mathematically it is stated as : A statement p(n), involving natural numbers n is true for all natural numbers n if we can show that ,

1. The statement p(n) is true when n = 1.
2. The statement p(n) is true when n = k+1,assuming that the statement is true for some natural number k .

The proof of statement p(n) using the first principle of mathematical induction consists of two steps:

1.The basis step: In which we show that the statement holds when n = 1.

2.The inductive step: In which we show that ***if*** the statement holds for n = k, ***then*** it also holds for n = k + 1.

The proposition following the word "if" in the inductive step is called the induction hypothesis. To perform the inductive step, we assume the induction hypothesis and then use this assumption to prove the statement for n = k + 1.

Examples:

1. Suppose we wish to prove the statement that: "**The addition of the first n natural numbers is equal to (n (n+1))/2 ,for all natural numbers n.**"

Proof : The proof that this statement is true for all natural numbers *n* proceeds as follows:

W*e* denote the statement to be proved by p(n).

So p(n) : $1 + 2 + 3 + \ldots + n = n(n+1)/2$

Basis of Induction: determine whether p(n) is a true statement for *n* = 1.

The sum of 1 and no other number is simply 1. And also $1(1 + 1) / 2 = 1$.

So the statement is true for *n* = 1.

Inductive step :Now we have to show that if the statement holds when *n* = k, then it also holds when *n* = k + 1.

Assume the statement is true for *n* = k, i.e. assume that, p(k) is true.

i.e. $1 + 2 + 3 + \ldots + k = k(k+1)/2$

Add (k + 1) which is the left-hand side's next term, to both sides. This does not change the equality:

$$\therefore 1 + 2 + 3 + \ldots + k + (k + 1) = [\,k(k+1)/2\,] + (k + 1)$$
$$= [k\,(k+1) + 2\,(k + 1)]/2$$

$$\therefore 1 + 2 + 3 + \ldots + k + (k + 1) = [\,(k+1)\,(k + 2)]/2$$

So, p(k +1) is proved assuming p(k) to be true.

Hence this statement p(n) is true for all natural numbers n because of the first principle of mathematical induction.

2. Prove the following formula for the sum of consecutive cubes:

$\mathbf{1^3 + 2^3 + 3^3 + \ldots + n^3 = [n^2 (n+1)^2] / 4}$, for all natural numbers n.

Proof : We denote the statement to be proved by p(n).

$\therefore$ p(n) : $1^3 + 2^3 + 3^3 + \ldots + n^3 = [n^2 (n+1)^2] /4$

Basis of Induction: determine whether p(n) is a true statement for $n = 1$.

The sum of 1^3 and no other number is simply 1.

And also $1^2 (1 + 1)^2 / 2 = 1(2)^2/4.$

$= 4 / 4$

$= 1$

So the statement is true for n = 1.

Inductive step : Assume the statement is true for $n = k$, i.e. assume that, p(k) is true.

p(k) : $1^3 + 2^3 + 3^3 + \ldots + k^3 = [k^2 (k+1)^2] /4$

Then we have to show that the statement is true for its successor, k + 1.

$\therefore$ Add $(k + 1)^3$ which is the left-hand side's next term, to both sides of p(k). This addition does not change the equality.

$$\begin{aligned} 1^3 + 2^3 + 3^3 + \ldots + k^3 + (k + 1)^3 &= [k^2 (k+1)^2] /4 + (k + 1)^3 \\ &= [(k+1)^2 (k^2 + 4 (k + 1))] / 4 \\ &= [(k+1)^2 (k^2 + 4k + 4)] / 4 \\ &= (k+1)^2 (k+ 2)^2) / 4 \end{aligned}$$

$\therefore 1^3 + 2^3 + 3^3 + \ldots + k^3 + (k + 1)^3 = (k+1)^2 (k+ 2)) / 4$

So, p(k +1) is proved by assuming that p(k) to be true.

Hence this statement p(n) is true for all natural numbers n, by the first principle of mathematical induction.

3. Prove that : The sum of the first n odd numbers is equal to the nth square:

i.e. $\mathbf{1 + 3 + 5 + 7 + \ldots + (2n - 1) = n^2}$, for all natural numbers n.

Basis of Induction: Determine whether p(n) is a true statement for $n = 1$.

On the left hand side , the sum of 1and no other number is simply 1.

And on the right hand side, $1^2 = 1.$

So the statement is true for n = 1.

Inductive step : Assume the statement is true for $n = k$, i.e. assume that, p(k) is true.

p(k) : $1 + 3 + 5 + 7 + \ldots + (2k - 1) = k^2$

Then we have to show that the statement is true for k + 1.

$\therefore$ Add [2(k+1) – 1] which is the left-hand side's next term, to both sides of p(k). This addition does not change the equality.

$\therefore 1 + 3 + 5 + 7 + \ldots + (2k - 1) + [2(k+1) - 1] = k^2 + [2(k+1) - 1]$
$\therefore 1 + 3 + 5 + 7 + \ldots + (2k - 1) + 2k + 1 \quad = k^2 + [2k+1]$
$= (k + 1)^2$

This is the required statement p(k+1).
So, p(k +1) is proved by assuming that p(k) is true.

$\therefore$ $1 + 3 + 5 + 7 + \ldots + (2n - 1) = n^2$ is true for all natural numbers n, by the principle of mathematical induction.

4. Prove that :
$1 + 4 + 9 + \ldots + n^2 = n (n + 1) (2n + 1) / 6$ for all positive integers n.

Proof : Here the statement to be proved is ,
p(n) : $1^2 + 2^2 + 3^2 + \ldots + n^2 = n (n + 1) (2n + 1) / 6$.
<u>Basis of Induction:</u> Determine whether p(n) is a true statement for $n = 1$.
On the left hand side, the sum of 1^2 and no other number is simply 1.
And on the right hand side, value is,

$$1 (1 + 1) (2\times1 + 1) / 6 = (1 \times 2 \times 3) /6 = 6 / 6 = 1.$$

So the statement is true for n = 1.
<u>Inductive step</u> : Assume the statement is true for $n = k$, i.e. assume that, p(k) is true.
p(k) : $1^2 + 2^2 + 3^2 + \ldots + k^2 = k (k + 1) (2k + 1) / 6$.
Then we have to show that the statement is true for k + 1.
$\therefore$ Add [$(k+1)^2$] which is the left-hand side's next term, to both sides of p(k). This addition does not change the equality.

$$\therefore 1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 = [(k) (k + 1) (2k +1) / 6]+(k+1)^2$$

$$= [(k + 1) (k + 2) (2k +3)] / 6.$$

This is the required statement p(k+1).
So, p(k +1) is proved by assuming that p(k) is true.

$\therefore$ $1^2 + 2^2 + 3^2 + \ldots + n^2 = n (n + 1) (2n + 1) / 6$, is true for all positive integers n, by the principle of mathematical induction.

5. Prove that : $\boldsymbol{n!} \geq 2^{n-1}$ for all positive integers $n \geq 1$.
$n! = n\times (n - 1) \times (n - 2) \times (n - 3) \times\ldots\times 3 \times 2 \times 1$.
Proof : Here the statement to be proved is ,
p(n) : $n! \geq 2^{n-1}$.
<u>Basis of Induction:</u> Determine whether p(n) is a true statement for $n = 1$.
On the left hand side, $n! = 1! = 1$.

And on the right hand side, $2^{n-1} = 2^{1-1} = 2^0 = 1$.
So the statement is true for n = 1.
Inductive step : Assume the statement is true for n = k, i.e. assume that, p(k) is true, where p(k) : $k! \geq 2^{k-1}$.
Then we have to show that the statement is true for k + 1.
∴ Multiply both sides of the inequality by (k+1) which is the left-hand side's next term, to both sides of p(k). This multiplication does not change the inequality.

$$\therefore (k+1) \times k! \geq (k+1) \times 2^{k-1}$$

$$\therefore (k+1)! \geq (k+1) \times 2^{k-1}$$

$$\geq 2 \times 2^{k-1} \quad \text{.... because } k+1 \geq 2 \text{ as } k \geq 1.$$

$$= 2^{(k+1)-1} = 2^k$$

$\therefore (k+1)! \geq 2^{(k+1)-1}$. This is the required statement p(k+1).

So, p(k +1) is proved by assuming that p(k) is true.
∴ $n! \geq 2^{n-1}$, is true for all all positive integers n ≥ 1, by the principle of mathematical induction.

6. Prove that : **$7^n - 1$ is divisible by 6** for all natural numbers n ≥ 1.
Proof : Here the statement to be proved is ,
p(n) : $7^n - 1$ is divisible by 6.
Basis of Induction: Determine whether p(n) is a true statement for n = 1.
For n = 1,
$7^n - 1 = 7^1 - 1 = 6$, it is divisible by 6.
So the statement is true for n = 1.
Inductive step : Assume the statement is true for n = k,
i.e. assume that, $7^k - 1$ is divisible by 6.
∴ $7^k - 1 = 6m$, for some integer m.
∴ $7^k = 6m + 1$ --------------------------------(I)
Then we have to show that the statement is true for k + 1. i.e. we have to show that $7^{(k+1)} - 1$ is divisible by 6.

Now, $7^{(k+1)} - 1 = (7 \times 7^k) - 1$
$= (7 \times (6m + 1)) - 1$, using (I) above.
$= 42m + 7 - 1$
$= 42m + 6$, it is divisible by 6.

∴ $7^{(k+1)} - 1$ is divisible by 6 , if $7^k - 1$ is divisible by 6.

So, p(k +1) is proved by assuming that p(k) is true.

$\therefore$ $7^n - 1$ is divisible by 6 , for all positive integers $n \geq 1$, by the principle of mathematical induction.

- ***Self Test I*:**

Using principle of mathematical induction prove that the given statements are true for all natural numbers n.

1. $2 + 7 + 12 + \ldots + (5n - 3) = [n(5n - 1)]/2$.
2. Sum of the first n even natural numbers is $n(n+1)$.
3. $1(1!) + 2(2!) + 3(3!) + \ldots + n(n!) = (n+1!) - 1$.
4. $1 + 2 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 1$.
5. $1 + 7 + 7^2 + 7^3 + \ldots + 7^{n-1} = (7^n - 1)/6$.
6. $2n^3 + 9n^2 + 13n + 7 > 0$.
7. $n^2 + n$ is an even number.
8. $5^n - 1$ is divisible by 4.
9. $11^n - 6$ is divisible by 5.
10. $30 - 6(5^n)$ is divisible by 24.

3.3 Summary for Unit 3

In this unit learners studied the following topics in details:

1. The first principle of Mathematical Induction .
2. Applications of the first principle of Mathematical Induction to prove different mathematical statements about natural numbers.