UNIT 15 : Introduction to Graph Theory

15.0 : Unit Objectives:

By the end of this Unit, learners should be able to:

- Understand the concept of a graph in discrete mathematics.
- Understand common terminology in Graph theory.
- Use different representation of graphs.
- Explain different types of graphs.
- Describe Eulerian and Hamiltonian graphs.
- Describe Planar graphs and colouring problem.
- Define and use trees.

15.1 : Unit Introduction:

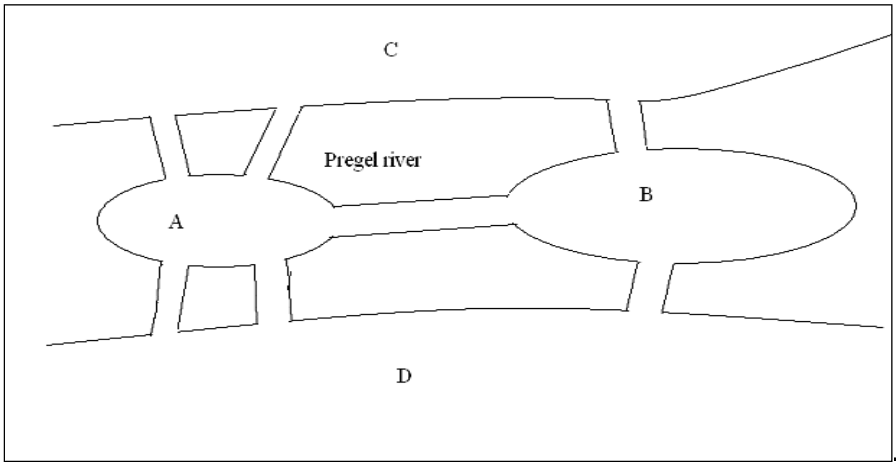
Graph theory is one of the widely used branches of mathematics. Although the first paper in graph theory was published in 1736 A.D. there has been widespread and intense interest in this subject since 1920s. One of the reasons for the recent interest in graph theory is its applicability in many diverse fields including computer science, chemistry, geography, electronics, electrical engineering etc. It is because one can represent a highway map of some region using a graph or even an electronics network can also be represented by a graph.

It is known that The first problem solved in graph theory was the "Seven bridges problem of Königsberg." The paper about solution to this problem was presented by well known mathematician Leonhard Euler in 1736 A.D. So he is also referred as the" father of graph theory."

15.2 : Graph :

In the city of Königsberg (now called Kaliningrad in Russia), two islands in the Pregel river were connected to each other and the river banks by seven bridges. See a crude map drawn of this geographical topology in the figure1.

The puzzle related with this geographical topology was that: "Starting from any one of the four land portions A, B, C and D is it possible to walk over each bridge exactly once and return to the starting point?" Leonard Euler in his paper explained that it is not possible to do this. He represented the map of this geographical topology in a diagrammatic way, which is now termed as a graph.



The graph representation of above map of Königsberg is as in figure 3 below. In this diagram or a graph the land portions A,B,C,D are denoted by dots or small circles called vertices and the bridges were denoted by arcs named e_1, e_2, e_3, e_4, e_5, e_6 and e_7 called as the edges in the graph.
Observe that how in the graph the land portions are represented by vertices i.e. the points and the bridges are represented by edges or arcs.

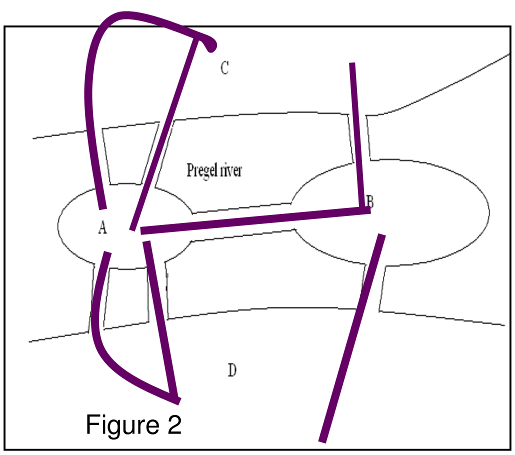


Figure 2

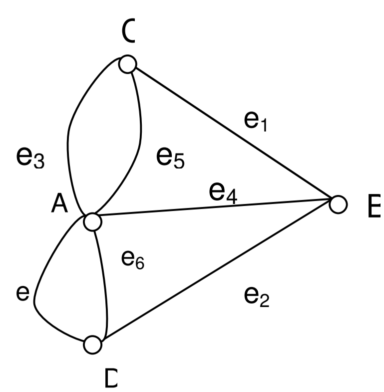


Figure 3**: Representation of Seven bridges problem**

We will now study the formal definition of a graph.

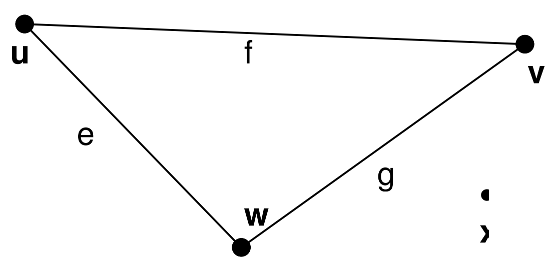
Definition 15.2.1: <u>Graph:</u> A graph G consists of a non empty set V of vertices and a set E of edges such that each edge is associated with a pair of vertices. We denote such graph as G = (V, E).

Every graph can be represented diagrammatically in which vertices are represented by points or dots and the edged are represented by lines or arcs joining the corresponding points i.e. vertices.

Examples:

- Let G = (V,E)be a graph where the set of vertices, V = { u, v, w,x } and the set of edges, E = { e, f, g }. Where the edge e represents the pair (u, w) or (w, u), f represents the pair (u, v) or (v, u), g represents the pair (v, w) or (w, v),Note that the pairs in set E are unordered.

This graph G can also be represented by the diagram given below.



- The graph representing seven bridges problem in figure 3 above is a graph G = (V, E); where the set of vertices = V = {A, B, C, D}, and the set of edges, E = {(A, B), (A, C), (A, C),(A, D), (A, D), (B, C),(B, D)}.

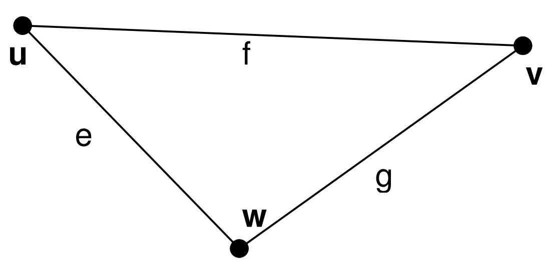
We can write this edge set also as E = {e_1, e_2, e_3, e_4, e_5, e_6, e_7}

where e_1= (B, C) or (C, B) , e_2 = (B, D) or (D, B), e_3 = (A, C) or (C, A) , e_4 = (A, B) or (B, A), e_5 = (A, C) or (C, A), e_6 = (A, D) or (D, A) and e_7= (A, D) or (D, A).

15.3 : Commonly used terminology in graph theory:

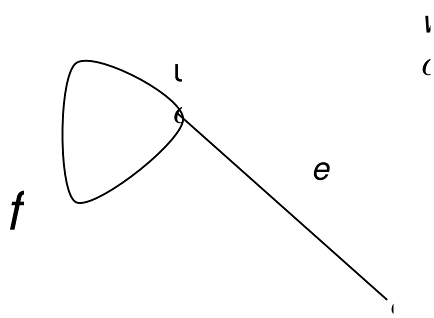
1. **Incidence and adjacency :** If in a graph an edge e is associated with the unordered pair of vertices *u* and *v* then , the edge e is said to be incident on *u* and *v*, and the vertices *u* and *v* are also said to be adjacent vertices.

Examples:

- 

In the graph given above the edge f is incident on the vertices u and v, so u and v are adjacent vertices, the edge e is incident on the vertices u and w so u and w are adjacent vertices and the edge g is incident on the vertices v and w so v and w are adjacent vertices. So all vertices are adjacent to each other.

- In the graph G_1 below the edge e is incident on the vertices u and v, so u and v are adjacent vertices and the edge f is incident on the vertex u only. Such edge is called a loop. But in this graph u and w are not adjacent vertices as there is no edge from vertex u to the vertex w. In fact w is not adjacent to any other vertices.



G_1

<u>2**. Degree of a vertex**</u>: The degree (or valency) of a vertex is the number of edges incident at that vertex. Degree of a vertex v is denoted as d(v).

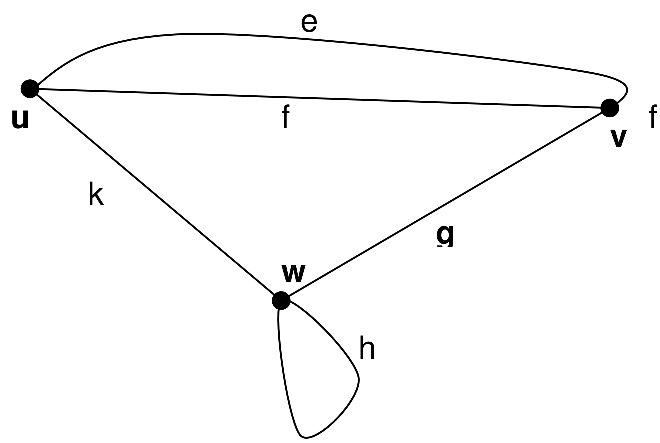
A loop contributes 2 to the degree of that vertex on which it is incident.

Examples:

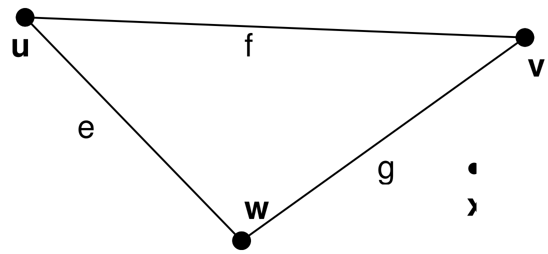
- In the graph drawn in figure on the right side , the degrees of vertices are

d(u) = 3 , d(v) = 3 and

d(w) = 4



- In the graph drawn in figure on the right side , the degrees of vertices are ; d(u) = 2 , d(v) = 2 , d(w) = 2 and d(x) = 0.

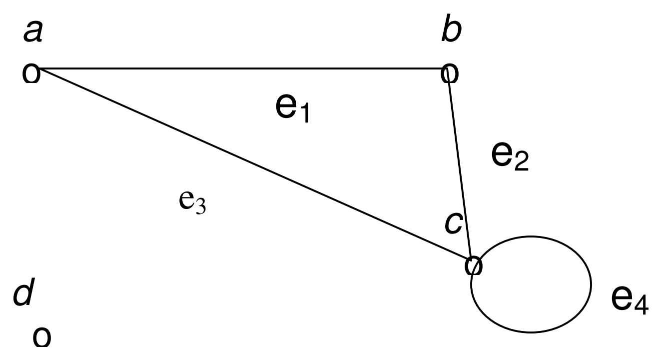


3. Self Loop and parallel edges:

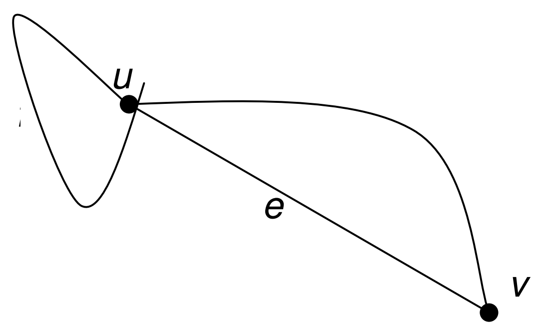
If in a graph an edge is incident on a single vertex then it is called as a loop. The edges which are incident on the same pair of vertices are called parallel edges.

Examples:

- In the graph drawn in figure on the right side , the edge e_4 is incident on the vertex c only . So the edge e_4 is a loop.



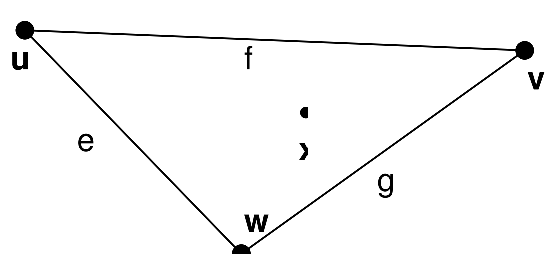
- In the graph drawn in the figure on the right side , the edge f is incident on the vertex u only . So the edge f is a loop. In this graph the edges c and e are incident on the same pair of vertices i.e. u and v so these are parallel edges.



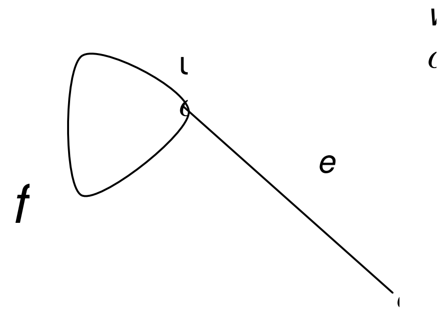
4. Isolated vertex: A vertex which is not incident on any edgei.e. a vertex which is not adjacent to any other vertex is called an isolated vertex in a graph.

Examples:

- In the graph drawn in figure on the right side , the vertex x is an isolated vertex



- In the graph drawn in figure on the right side , the vertex w is an isolated vertex



G_1

5. **Path in a graph:** Let u_0 and u_n are vertices in a graph G, a path P from vertex u_0 to vertex u_n is an alternating sequence of edges and vertices beginning with vertex u_0 and ending with vertex u_n i.e. P: $u_0\ e_1\ u_1\ e_2\ u_2\ e_3\ u_3\ e_4 \ldots\ e_n\ u_n$ such that every edge in this sequence is incident on preceding and succeeding vertices.

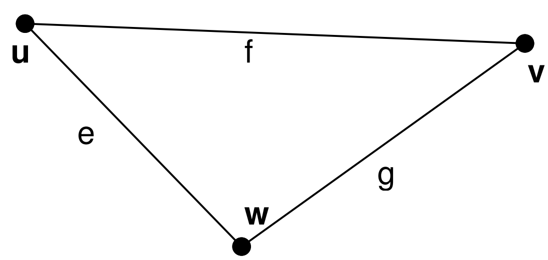
Note :

1. A path is of length n if it contains n edges.
2. If in a path no vertices are repeated then it is called a simple path.

Examples:

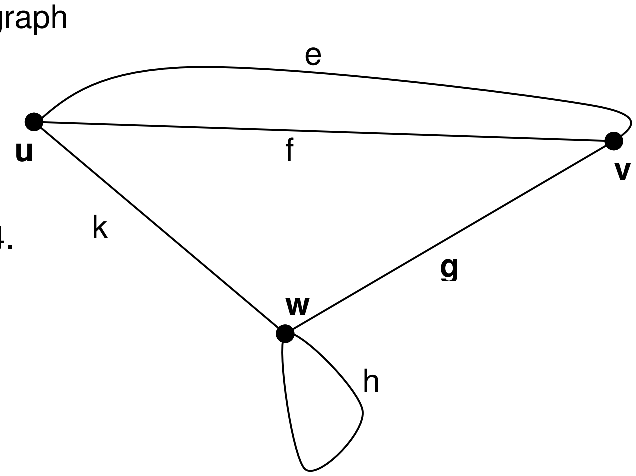
- Two of the paths in the graph drawn in figure on the right side are ,

P : u e w g v , it is u to v path of length 2
Q: u f v g w e u , it is path of length 3.
Both are simple paths



- Following are some paths in the graph drawn in figure on the right side ,

P : u k w h w g v ,
it is u to v path of length 3
Q: u f v e u k w g v, it is path of length 4.
Both are not simple paths .

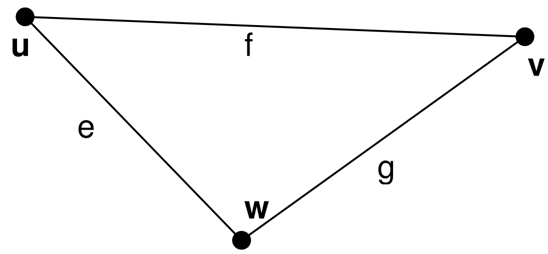


6. **Cycle in a graph:** A cycle (or circuit) in a graph is a path of non zero length from a vertex u to u with no repeated edges.

Examples:

- A cycle in the graph drawn in figure on the right side is,

C: u f v g w e u

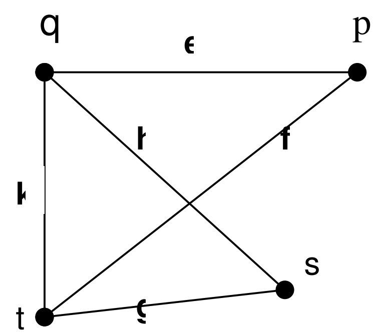


- Some cycles in the graph drawn in figure on the right side are,

C_1: p e q k t f p

C_2: q k t g s h q

C_3: p e q h s g t f p



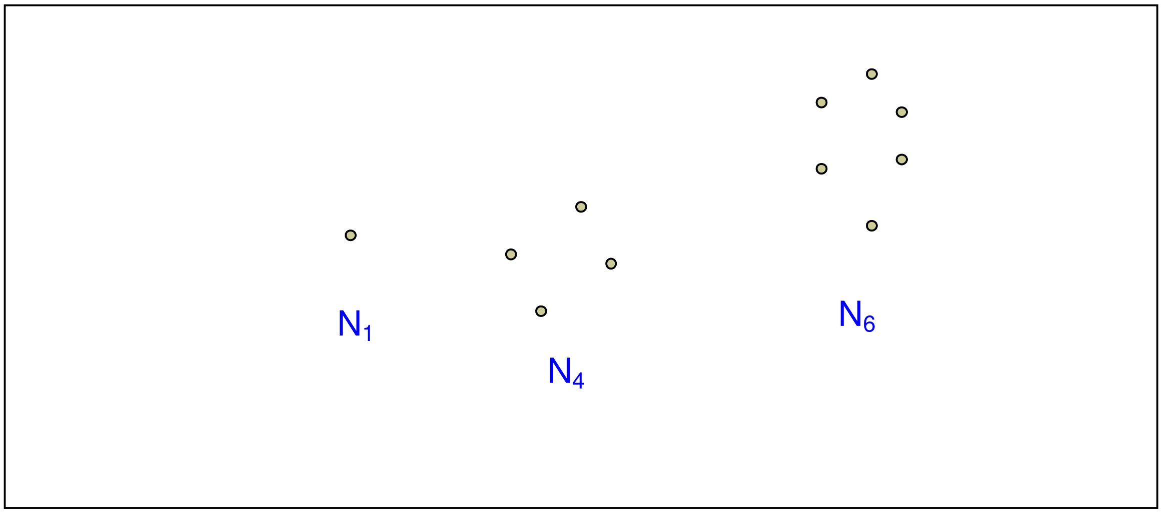
Self Test I:

15.4 : Some important types of graphs :

1:**Definition 15.4.1**: Null graph: The graph containing no edges is called a null graph. A null graph containing n vertices is denoted as N_n.

Examples:

- The null graphs N_1, N_4, and N_6 are null graphs containing 1, 4 and 6 vertices respectively , these null graphs can be drawn as in following figure.

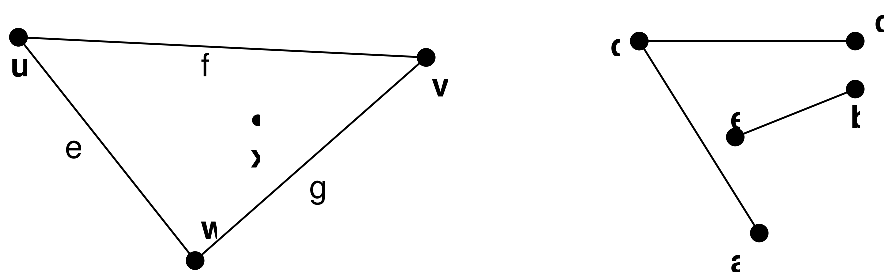


2: **Definition 15.4.2:**Simple graph: A graph which contains neither loops nor parallel edges is called a simple graph.

Examples:

- The graphs represented by the 2 figures below are simple graphs but the graph representing seven bridges problem mentioned earlier is not a simple graph.

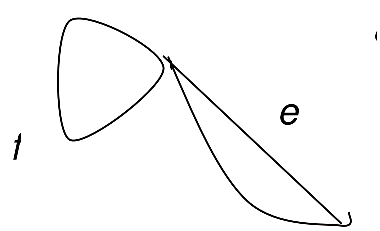
Note that a null graph is a simple graph.



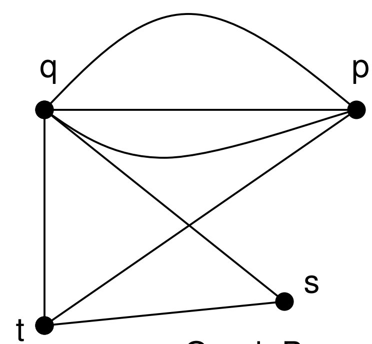
3. **Definition 15.4.3:** Multigraph or multiple graph: A multigraph is a graph containing multiple or parallel edges and /or loops between the same vertices. The graph in figure A is a multigraph But the graph in Figure B is a simple graph.

Examples:

- The following graphs, the graph A and the graph B below are multiple graphs. In the graph A the edge f is a loop and the edges e and g are parallel edges. And in the graph B there are 3 parallel edges incident on the pair p and q of vertices.



Graph A

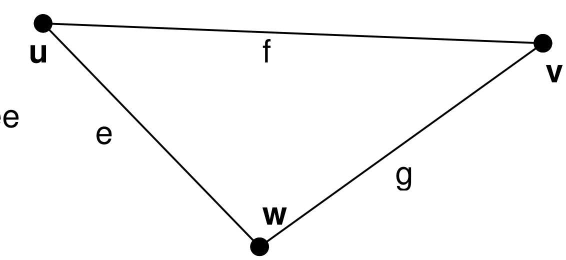


Graph B

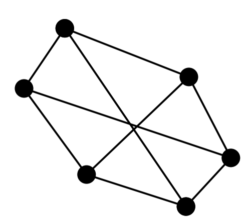
4. **Definition 15.4.4:** Regular graph: A regular graph is a graph, in which all of the vertices have the same degree. i.e. in a regular graph the number of edges incident at every vertex is the same. If each vertex has degree r then that regular graph is called as r-regular graph.

Examples:

- In the graph drawn in figure on the right side d(u) = 2, d(v) = 2, d(w) = 2 , So it is a regular graph of degree two or a 2-regular graph.



- In the graph drawn in figure on the right side is a regular graph of degree 3 or a 3- regular graph, as each vertex of this graph has degree three.



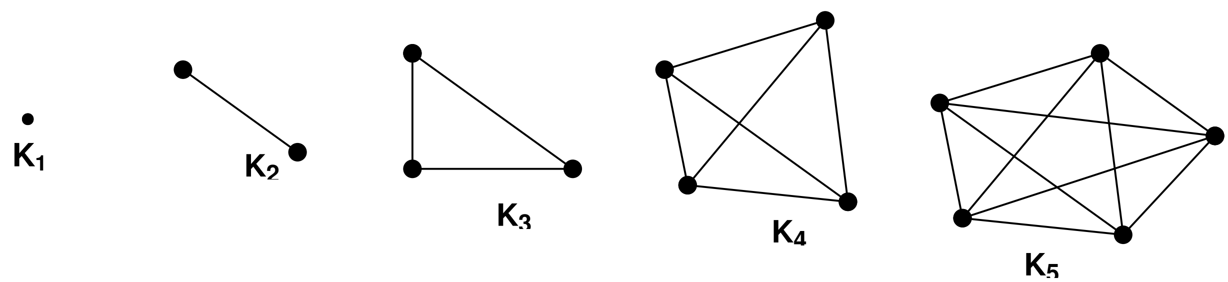
- Null graph is a regular graph of degree 0.

5. Definition 15.4.5:<u>Complete graph</u> :The **complete graph** on p vertices denoted by K_{p}, is a simple graph with p vertices in which there is an edge between every pair of distinct vertices.

Examples:

- The complete graphs K_1 containing 1 vertex is just a vertex without any edge.
- K_2 is a complete graph on 2 vertices and K_3 is a complete graph on3 vertices,

K_4 is a complete graph on 4 vertices and K_5 is a complete graph on 5 vertices. These are as shown in the figure below.



Note that all complete graphs are regular graphs. In fact a complete graphs K_p is a (p–1) regular graph because every vertex in this graph is adjacent to remaining p – 1 vertices.

Self Test II:

15.5 : Representation of Graph using Matrix :

1. Definition 15.5.1: Adjacency matrix**:** Let G be a graph with n vertices which are ordered as $v_1, v_2, v_3, v_4, \ldots, v_n$. Then the Adjacency matrix of the graph G is a matrix $A = [a_{ij}]$ of order $n \times n$, defined by

$$a_{ij} = \text{the number of edges incident on the adjacent vertices } v_i \text{ and } v_j \text{ ,}$$

and $a_{ij} = 0$, when vertex v_i and the vertex v_j are not adjacent.

Note that if the graph contains loops incident on a vertex v_i then

$$a_{ii} = \text{ twice the number of loops incident on the vertex } v_i \text{ .}$$

Examples:

- The adjacency matrix of the graph in the figure 1 on the right side is,

$$A = \begin{array}{c} u \\ v \\ w \end{array} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This because there are 2 edges i.e.
a loop in this case from vertex u to u and
1 edge from vertex u to vertex v and vice versa.

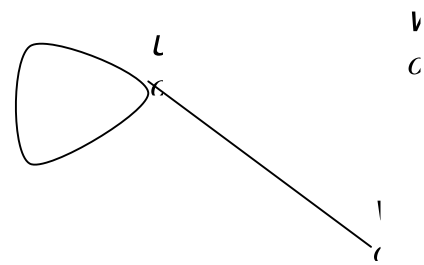


Figure 1:

- The adjacency matrix of the graph in the figure 2 on the right side is,

$$A = \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 2 & 1 \\ 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

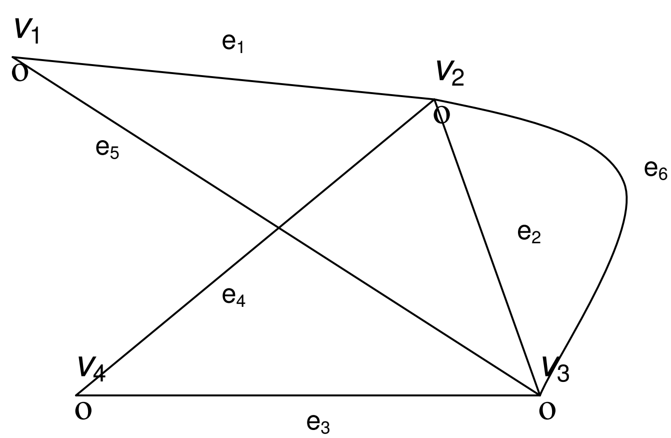


Figure 2 :

2. Definition 15.5.2: Incidence matrix :

Let G be a graph with n vertices which are ordered as $v_1, v_2, v_3, v_4, \ldots, v_n$ and m edges which are ordered as $e_1, e_2, e_3, e_4, \ldots, e_m$.

Then the Incidence matrix of the graph G is a matrix $A = [a_{ij}]$ of order $n \times m$, defined by

$$a_{ij} = 1, \text{ if vertex } v_i \text{ is incident on the edge } e_j$$
$$\text{and } a_{ij} = 0, \text{ otherwise.}$$

Examples:

- The incidence matrix of the graph in the figure 1 on the right side is,

$$A = \begin{array}{c} u \\ v \\ w \end{array} \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

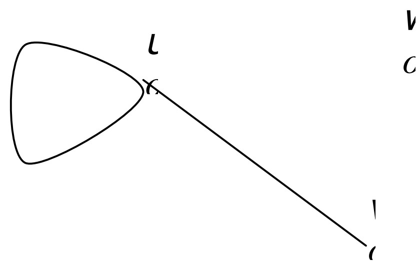


Figure 1:

In this matrix the first column corresponds to the edge (u ,u) and there are 2 edges i.e. a loop from vertex u to u. And the second column corresponds to the edge (u ,v), so we write 1 in the rows corresponding to vertex u and vertex v.

- The incidence matrix of the graph in the figure 2 on the right side is,

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccccc} e1 & e2 & e3 & e4 & e5 & e6 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

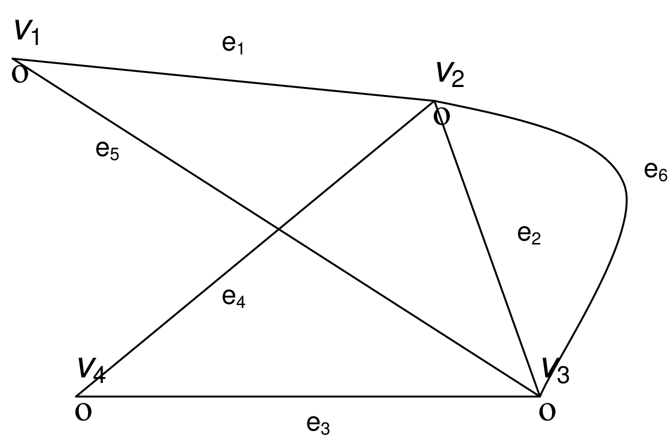


Figure 2 :

15.6 : Eulerian and Hamiltonian graphs :

Definition 15.6.1: Connected graph: A graph G is connected if for every pair of vertices u and v in G , there is a path from vertex u to vertex v.

Other wise the graph G is disconnected.

Examples:

- The graph shown in the figure 2 is a connected graph because for every pair of vertices in this graph, there is a path from first vertex to the other vertex.
- The graph shown in the figure 1 is a disconnected graph because for the pair of vertices u and w in this graph, there is no path from vertex u to the other vertex w .In fact there is no path from the vertex w to any other vertex.
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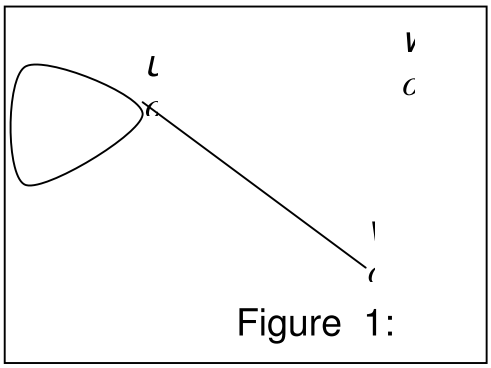


Figure 1:

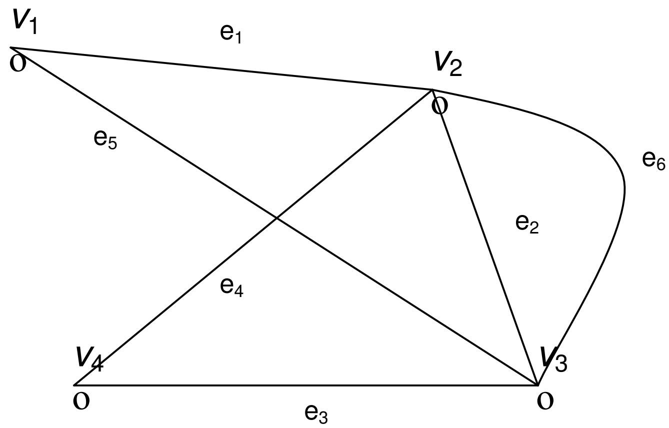


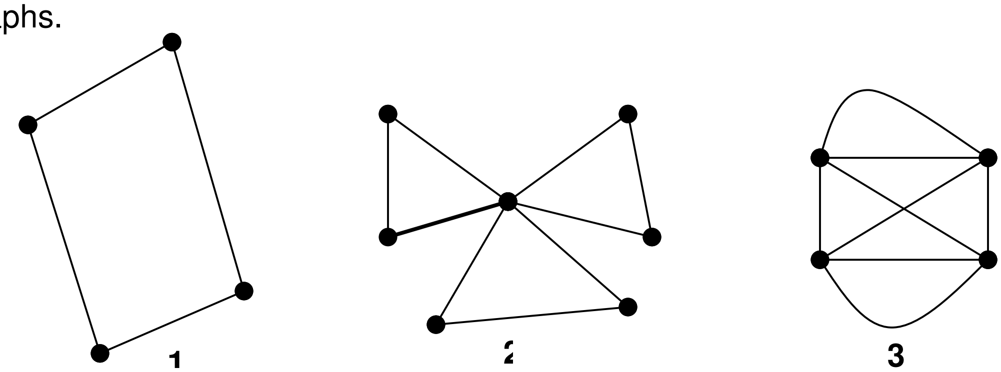
Figure 2 :

Definition 15.6.2: <u>Eulerian graph</u>**:** A graph is called as Eulerian graph if it contains a cycle which includes all of the edges and all of the vertices.

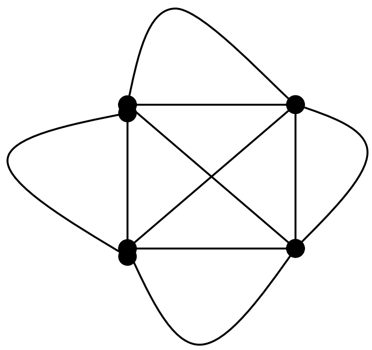
Note that, an Eulerian graph is always a connected graph in which all vertices have even degree.

Examples:

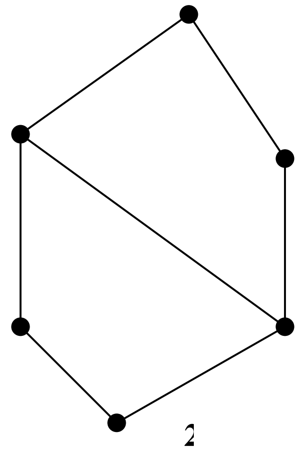
- The complete graphs K_3, K_5, K_7 etc are Eulerian graphs. In general complete graph K_n is a Eulerian graph when n is odd.
- The 3 graphs shown in the following figure are all Eulerian graphs.



- The 2 graphs shown in the following figure are not Eulerian graphs.

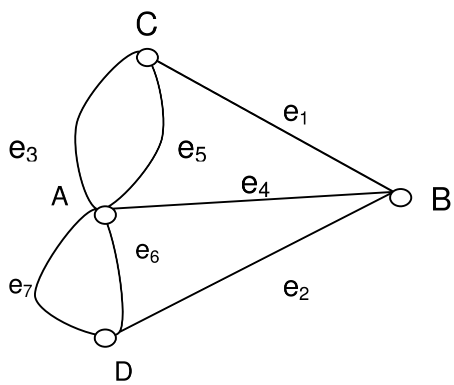


1



2

- The important example of non Eulerian graphs is the graph which represents seven bridges problem, drawn below.



Representation of Seven bridges problem

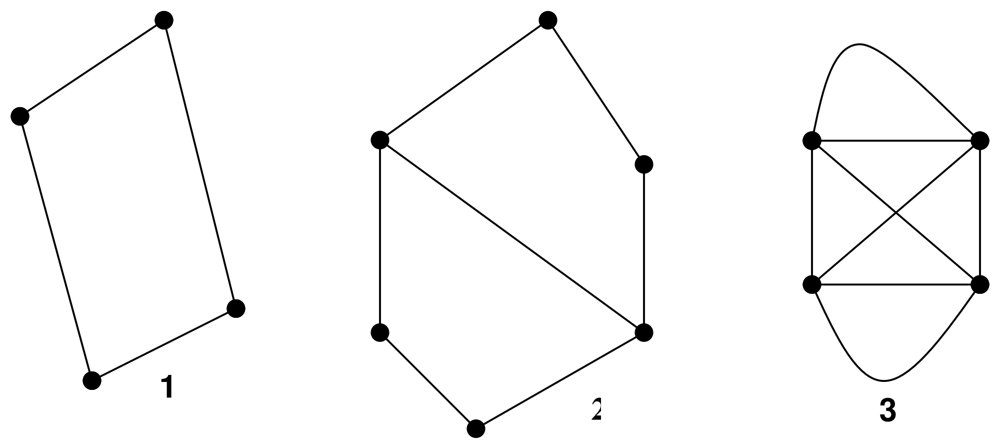
Definition 15.6.3: Hamiltonian graph**:** A graph G is called as Hamiltonian graph if it contains a cycle which includes every vertex of G exactly once, except that for the starting and ending vertex that appears twice.

Note that, a Hamiltonian graph is always a connected graph but any necessary and sufficient condition for a graph to be Hamiltonian is not known.

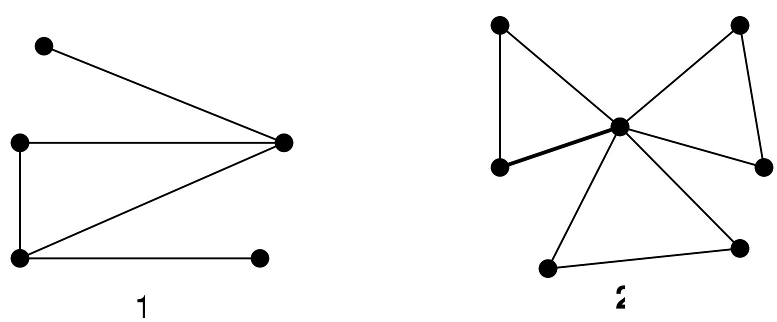
Examples:

- All complete graphs K_n are Hamiltonian graphs, for $n \geq 3$. As each complete graph includes at least one cycle in which every vertex other than initial vertex is appearing only once.

- The 3 graphs shown in the following figure are all Hamiltonian graphs.



- The 2 graphs shown in the following figure are not Hamiltonian graphs.

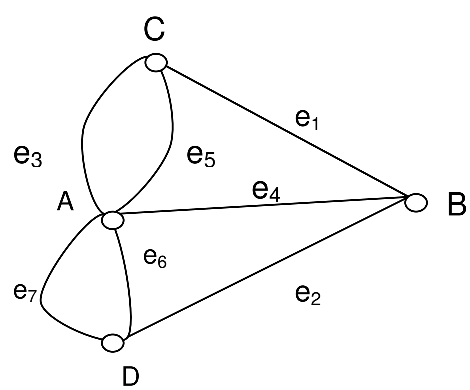


- The graph which represents seven bridges problem, drawn below is an example of a Hamiltonian graphs.

  Some Hamiltonian cycles in this graph are :

  C_1: A e_6 D e_2 B e_1 C e_5 A

  $C2_1$: A e_7 D e_2 B e_1 C e_3 A



Representation of Seven bridges problem

Self Test III:

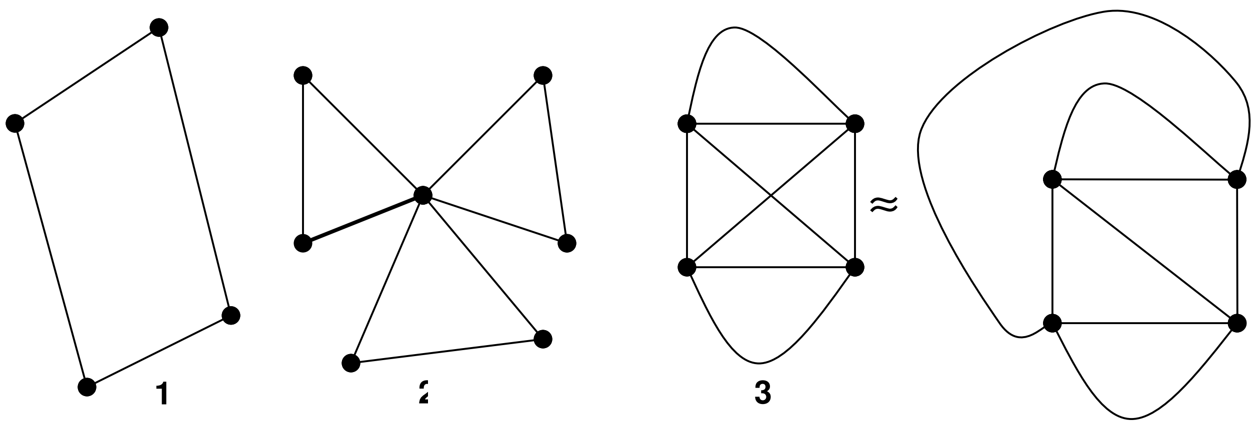
15.7 : Planar graphs and colouring problem:

15.7.1 : Planar graphs:

Definition 15.7.1: Planar graph**:** A graph is called as a planar graph if it can be drawn on a plane in such a way that no edges crass each other (except , of course at common vertices).

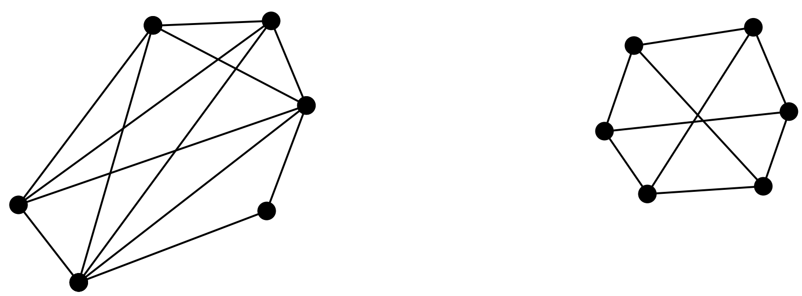
Examples:

- The complete graphs K_1, K_2, K_3 and K_4 are planar graphs. But other complete graphs K_n are not planar graphs, for $n \geq 3$.
- The 3 graphs shown in the following figure are all planar graphs. The 3rd graph in this diagram can be drawn in a different way and the crossing of edges can be avoided.



1 2 3 ≈ 3

- The 2 graphs shown in the following figure are all non planar graphs. These graphs in this diagram can not be drawn in a different way such that the crossing of edges can be avoided.



15.7.2 : Colouring problem:

Definition 15.7.2: Colouring of a graph**:** Colouring of a graph means to assign one or more distinct colours to the vertices of a graph in such a way that no two adjacent vertices are assigned the same colour.

The idea of colouring of planar graphs is useful in solving problems in day to day life such as timetable problem.

A planar graph can be used to represent a map also, hence colouring a map is equivalent to colouring a planar graph. Lots of research is done about colouring in graph theory. Now it has been proved by the four colour theorem that every simple planar graph can be coloured with four colours .

15.8 : Trees:

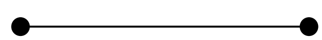
Trees form one of the most widely used subclass of graphs. There are many applications of trees. Trees are useful in computer science to organize data in a database. There are many equivalent definitions of a tree. We will see the simplest form of it.

Definition 15.8.1: Tree**:** A tree is a simple, connected graph in which no cycle exists.

Examples:

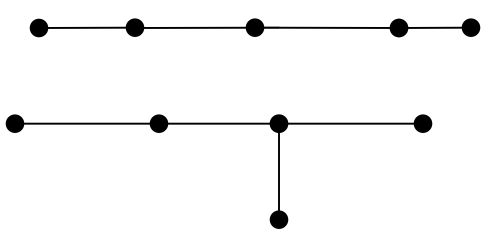
- The complete graphs K_1 and K_2 drawn below are trees, because these are connected graphs without any cycles. But other complete graphs K_n are not trees, for $n \geq 3$.

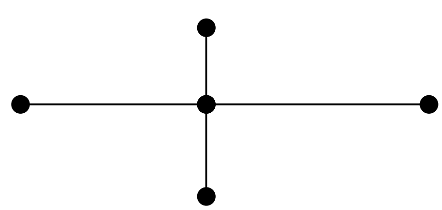
K_1



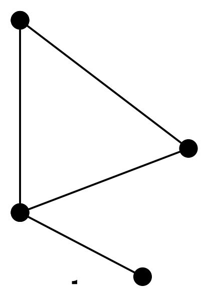
K_2

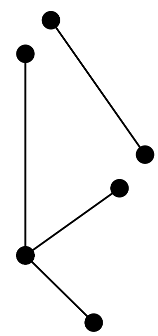
- The following 3 graphs are non isomorphic trees on 4 vertices





- The following 2 graphs are not trees because the first graph includes a cycle and the second graph is not a connected graph.





15.8.1 : Properties of trees

1. A tree containing n vertices , has n – 1 edges.
2. There exists unique path between any two vertices of a tree.
3. If any two nonadjacent vertices in a tree are joined by an edge, then the resultant graph contains exactly one cycle.

Self Test IV:

15.9 Summary for Unit 15:

In this unit learners studied the following topics in details:

1. The "Seven bridges problem of Königsberg."
2. The concept a graph and its different representations.
3. Different terms used in the graph theory.
4. Different types of graphs such as, null graph, simple graph, multigraph, regular graph and complete graph etc.
5. Adjacency matrix and incidence matrix.
6. Connectivity , Eulerian and Hamiltonian graphs.
7. Planar graphs.
8. Trees.